CALCULUS

INTRODUCTION

The calculus deals with infinitesimal changes in mathematical functions. It was invented in the 18th century more or less simultaneously by Isaac Newton and Gottfried Wilhelm Leibniz. Newton applied the calculus to the laws of motion and calculus has been the staple of mathematical physics ever since.

DIFFERENTIAL CALCULUS

Suppose we have a simple harmonic oscillator, an object which oscillates back and forth according to the function $x=2\sin(\pi t/4)$ where x and t are the position and time in mks units. It's easy to see that after 1 second the position of the object is $x=2\sin(\pi/4)=1.414214...$, but what is the velocity of the object at that time? We could approximate the velocity by calculating the distance Δx the object moves during a brief time increment Δt after 1 second. Then the velocity of the object after one second equals approximately $\Delta x/\Delta t$. The smaller the time increment, the more nearly $\Delta x/\Delta t$ will approximate the velocity at one second. The table below shows $\Delta x/\Delta t$ calculated with successively smaller values of Δt. The velocity appears to approach a value 1.11072 m/sec.

Δt	$x(t)$	$x(t+\Delta t)$	$\Delta x=x(t+\Delta t)-x(t)$	$\Delta x/\Delta t$
10	1.414214	1.414214	0.000000	0.000000
1	1.414214	2.000000	0.585786	0.585786
0.1	1.414214	1.520812	0.106598	1.065984
0.01	1.414214	1.425277	0.011063	1.106348
0.001	1.414214	1.415324	0.001110	1.110284
0.0001	1.414214	1.414325	0.000111	1.110677
0.00001	1.414214	1.414225	0.000011	1.110716

Table 1. Successive approximations to velocity $\Delta x/\Delta t$.

Calculating velocity numerically is very tedious, especially if, like Isaac Newton, you don't have a Macintosh Powerbook handy. But is there a better way? You bet, and the differential calculus provides it!

DIFFERENTIALS AND DERIVATIVES

A differential is an infinitesimal change in a variable. Mathematicians have rigorous and elegant ways of defining differentials which pretty much amount to saying that a differential is smaller than the smallest imaginable change, but not quite zero. A differential change in a variable x is notated $\mathrm{d}x$ and a small finite change in x is notated Δx. The "d" here is not a number multiplying x but rather a differential operator. A differential may be thought of as the limit of a finite change.

The derivative is another important concept in differential calculus. The derivative of a function $y(x)$ is the rate of change in the function as the variable x changes. It can be thought of as the quotient of the differential change in y, $\mathrm{d}y$, induced by the differential change in x, $\mathrm{d}x$. The differential is notated $\mathrm{d}y/\mathrm{d}x$. For example, the velocity of the simple harmonic oscillator discussed above would be written $\mathrm{d}x/\mathrm{d}t$.[1]

We can develop formulas for the differentials of any mathematical function. As an example, consider the function $y(x)=x^2$. The differential of y is given by

$$\mathrm{d}y=y(x+\mathrm{d}x)-y(x)=(x+\mathrm{d}x)^2-x^2=x^2+2x\mathrm{d}x+(\mathrm{d}x)^2-x^2=2x\mathrm{d}x+(\mathrm{d}x)^2=(2x+\mathrm{d}x)\mathrm{d}x.$$

But since the differential is so small, $\mathrm{d}x$ can be neglected in the last parentheses so that

$$\mathrm{d}x^2=2x\mathrm{d}x.$$

Expressions for the differentials of other functions may be derived by similar methods. Differentialss of several common functions are tabulated below.

$dk = 0$	where k is a constant
$dx^n = nx^{n-1}dx$	
$d(e^x) = e^x dx$	where e=2.71828…, the base of natural logarithms
$d(\ln x) = dx/x$	where $\ln x$ is the natural logarithm of x
$d(\sin x) = \cos x dx$	
$d(\cos x) = -\sin x\, dx$	

Table 2. Differentials of some common functions.

But how would we differentiate a function like $\cos x^2$ which isn't included in the table above? In such a case the chain rule

$$dy = (dy/du)du.$$

may be helpful. Using the chain rule

$$d(\cos x^2) = (-\sin x^2)(dx^2) = -2x \sin x^2 dx,$$

where we've let $u = x^2$. A direct consequence of the chain rule is that $d(ky(x)) = k dy(x)$ where k is a constant.

Three other useful relationships pertain to the differential of the sum, product, and quotient of two functions u and v. These are

$$d(u+v) = du + dv,$$
$$d(uv) = u dv + v du,$$
$$d(u/v) = (v du - u dv)/v^2.$$

EXAMPLES

SLOPE OF A CURVE

Figure 1 shows the plot of a function $y(x)$. The slope m of the curve at some point (x,y) is tan θ where θ is the angle tangent to the curve (the arrow in figure 1) makes with the x-axis. From the figure it can be seen that $m \cong \Delta y/\Delta x$ where $\Delta y=y(x+\Delta x)-y(x)$. As we make Δx (and hence Δy) smaller and smaller the approximation to m becomes better and better until it becomes perfect as $\Delta x \rightarrow \mathrm{d}x$. In (slightly) better mathematical terms we would write

$$m=\lim_{\Delta x, \Delta y \rightarrow 0} (\Delta y / \Delta x)=\mathrm{d}y/\mathrm{d}x.$$

Hence the slope of a curve $y(x)$ at a point (x,y) is simply $\mathrm{d}y/\mathrm{d}x$.

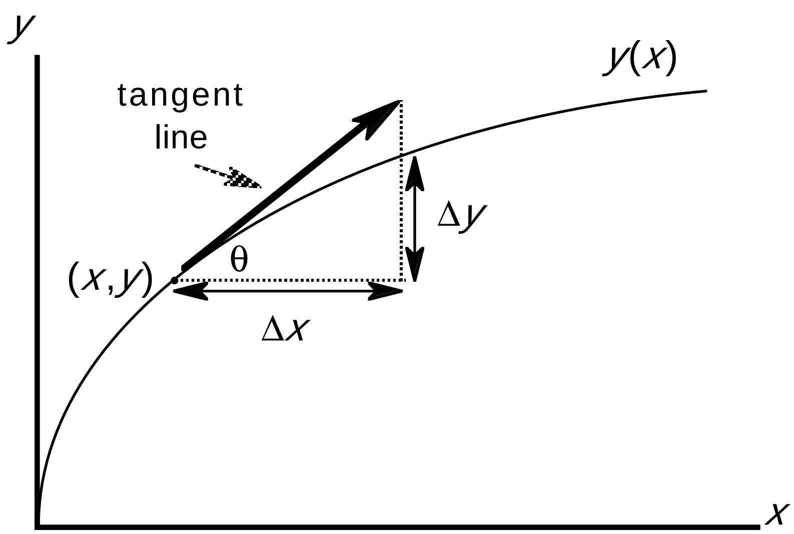


Figure 1. The slope of a function $y(x)$ at a point (x,y) equals the tangent of θ, the angle the tangent line at the point makes with the x-axis.

Example: What is the slope of the curve $y=5 \ln x$ when $x=1$?

Solution: The slope at any point is $m=\mathrm{d}y/\mathrm{d}x=5/x$, so when $x=1$, $m=5$.

MAXIMA AND MINIMA

A common problem is that of finding the greatest or least values of a function $y(x)$, its extrema. Figure 2 suggests how that may be done.

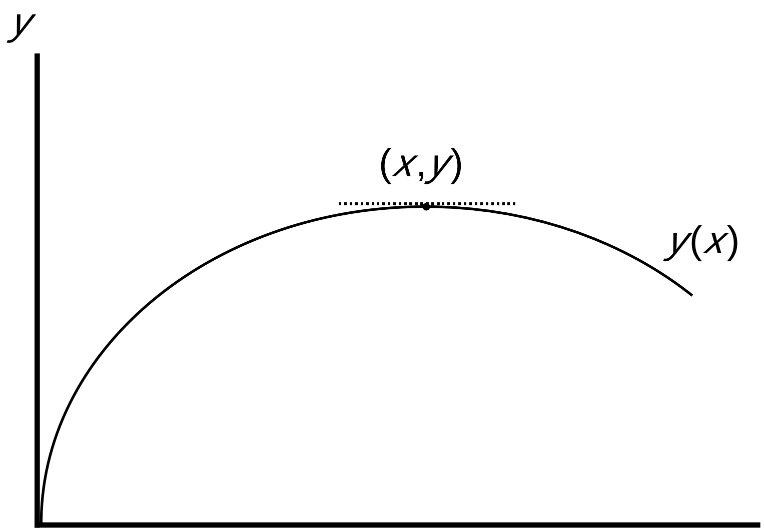


Figure 2. The slope of a function $y(x)$ vanishes at a maximum (x,y).

As may be seen, at a maximum the slope of the function is zero. Likewise the slope would be zero at a minimum (why?). So to find the extrema of a function, simply set the derivative equal to zero and solve for $y(x)$.

Example: Find the maximum of the function $4-x^2$.

Solution: The derivative of the function is $-2x$. This vanishes when $x=0$, so the maximum value of the function is 4.

KINEMATICS

Suppose we have an object whose position x varies with time t. The velocity v of the object is the time rate of change of x or $v=dx/dt$.

Example: The position of an object at time t is given by $x=1+2t-4t^2$. What is the velocity of the object when $t=2$?

Solution: The velocity of the object is $v=dx/dt=2-8t$. When $t=2$, the velocity is $v=2-8x2=-14$. The negative sign means the object is moving to the left.

Example: Let's reconsider the example which started this section, a simple harmonic oscillator back and forth according to the function $x=2\sin(\pi t/4)$ where x and t are the position and time in mks units. Find its velocity when $t=1$ second using differential calculus.

Solution: The velocity of the oscillator is $v=dx/dt=(\pi/2)\cos(\pi t/4)$, so when $t=1$ the velocity is $v=(\pi/2)\cos(\pi/4)=1.110721$ m/sec. This is the exact number which the numerical calculation approximated.

Example: The position of a simple harmonic oscillator at time t is given by $x=A\sin\omega t$ where A and ω are constants called the amplitude and angular frequency, respectively. What is the kinetic energy of the oscillator at time t if the mass of the osciallator is m.

Solution: The kinetic energy of an object is $E=mv^2/2$. The velocity of the simple harmonic oscillator is $v=dx/dt=A\omega\cos\omega t$, so the kinetic energy is $E=(mA^2\omega^2/2)\cos^2\omega t$. Note that the energy is proportional to the square of the amplitude, A.

INTEGRAL CALCULUS

Suppose we want to find the area of a quarter circle of unit radius. Of course we already know from the usual formula for the area of a circle that the answer is $\pi/4=0.785398\ldots$ But if we didn't know that we could calculate the area by plotting the circle which has equation $x^2+y^2=1$ in a Cartesian coordinate system and then find the area between the curve and the x- and y-axes, the shaded area in Figure 3. But how do we manage **that**?

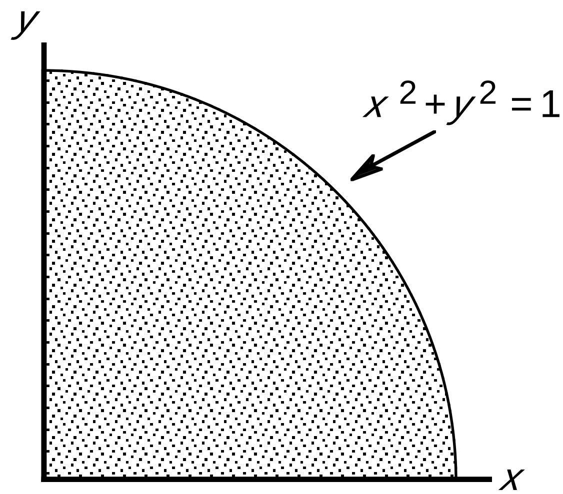


Figure 3. The area of a unit quarter circle is the area between the curve $x^2+y^2=1$ and the x- and y-axes, the shaded area of the diagram above.

One way would be to divide the area up into a number of rectangles as in

Figure 4. Each rectangle has width Δx, height y equal to the height of the curve at the center of the rectangle and has area $y\,\Delta x$. The total area under the curve is approximately equal to the sum of the rectangles.

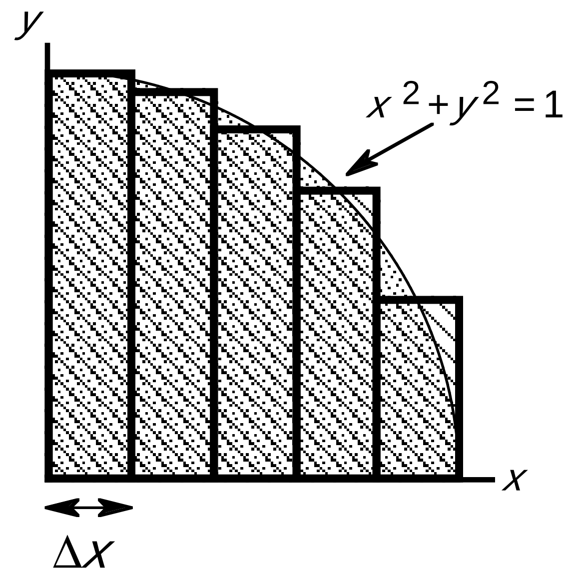


Figure 4. The area of a unit quarter circle may be approximated by summing the areas of the rectangles which straddle it in the diagram above.

The approximation of Figure 4 is carried out in Table 3. Clearly this is a crude approximation. We could improve the approximation by making Δx smaller and taking more rectangles. The effect of doing this is shown in Table 4.

x	$y(x)=\sqrt{[1-(x+\Delta x/2)^2]}$	$y(x)\Delta x$
0.0	0.995	0.199
0.2	0.954	0.191
0.4	0.866	0.173
0.6	0.714	0.143
0.8	0.436	0.087
	area$=\Sigma y(x)\Delta x=$	0.793

Table 3. Calculation of the area under a quarter unit circle using a five rectangle approximation. From geometry the correct answer is $\pi/4=0.785\ldots$

Δx	$\Sigma y(x)\Delta x$
0.20	0.793
0.10	0.788
0.05	0.786

Table 4. As smaller and smaller increments are used, the sum approximating the area of a quarter circle approximates better and better the answer $\pi/4=0.785\ldots$

If we let the interval Δx become infinitesimal, the sum approximating the area becomes exact and the summation becomes an integral. In symbols we'd write

$$\lim_{\Delta x \to 0}\left[\sum y(x)\Delta x\right] = \int y(x)\mathrm{d}x.$$

where $\sum$ is the summation sign and $\int$ the integral sign. We call $\int y(x)\mathrm{d}x$ the integral of the function $y(x)$. The area of the quarter circle in the example above would be given by

$$\text{area} = \int y(x)\mathrm{d}x = \int_0^1 \sqrt{1-x^2}\,\mathrm{d}x.$$

The numbers above and below the integral sign are called the limits of integration and indicate the range of values assumed by the variable x.

So how do we evaluate the integral of a function? The answer is provided by The Fundamental Theorem of Calculus which says that differentiation and integration are reciprocal operations; in other words the integral of the differential of a function **is** the function. In more formal terms, The Fundamental Theorem of Calculus can be stated two ways, as follows:

Statement 1:

$$\int_a^b \mathrm{d}y(x) = y(b) - y(a)$$

Statement 2:

$$\int \mathrm{d}y(x) = y(x) + C \quad \text{where } C \text{ is a constant.}$$

Statement 1 gives the definite integral of the function and Statement 2 gives the indefinite integral. We can use the fundamental theorem of calculus to evaluate integrals by reading Table 2 right to left instead of left to right. Suppose, for example, we want to evaluate $\int(3x^2)\mathrm{d}x$. From Table 2 we know that $\mathrm{d}(x^3)=3x^2$ so

$$\int(3x^2)\mathrm{d}x=x^3+C.$$

Using this approach we can write a table of integrals (Table 5) corresponding to the differentials of Table 2.

$\int x^n\mathrm{d}x = (x^{n+1})/(n+1)$	where $n\neq-1$
$\int(e^x)\mathrm{d}x = e^x$	where e=2.71828…, the base of natural logarithms
$\int(1/x)\mathrm{d}x = \ln x$	where $\ln x$ is the natural logarithm of x
$\int(\sin x)\mathrm{d}x = -\cos x$	
$\int(\cos x)\mathrm{d}x = \sin x$	

Table 5. Integrals of some common functions. A constant C can be added to the right hand side of each equation.

It's also helpful to recognize that from the definition of an integral as limit of sums it follows that $\int ky(x)\mathrm{d}x=k\int y(x)\mathrm{d}x$ where k is a constant

It's possible to differentiate **any** reasonable function[2], but it's not always possible to integrate a function, however reasonable, in closed form (a form giving the integral as an algebraic expression like those of Table 5.)

Mathematicians have spent much energy devising ingenious strategies for integrating functions. We'll only mention one here, the substitution of variables. In this method a new function $u(x)$ is introduced to reduce an integral to a fundamental form.

Example: Evaluate the integral $S = \int_1^2 \frac{x}{(1+x^2)^3} \mathrm{d}x$.

Solution: Define the new function $u(x)=1+x^2$. The differential of u is $\mathrm{d}u=2x\mathrm{d}x$. The integral now becomes

$$S = \frac{1}{2}\int_2^5 \frac{1}{u^3}\mathrm{d}u = \left[\frac{u^{-2}}{(-4)}\right]_2^5 = -\frac{1}{100} + \frac{1}{16} = \frac{21}{400}.$$

where the limits of integration have been changed since $u(1)=2$ and $u(2)=5$.

Example: Find the area of a quarter unit circle by evaluating the integral

$$\text{area} = \int_0^1 \sqrt{1-x^2}\,\mathrm{d}x.$$

Solution: This involves a trigonometric substitution. Introduce a new variable θ such that $\sin\theta = x$ and $\mathrm{d}x=\cos\theta\mathrm{d}\theta$. The integral becomes

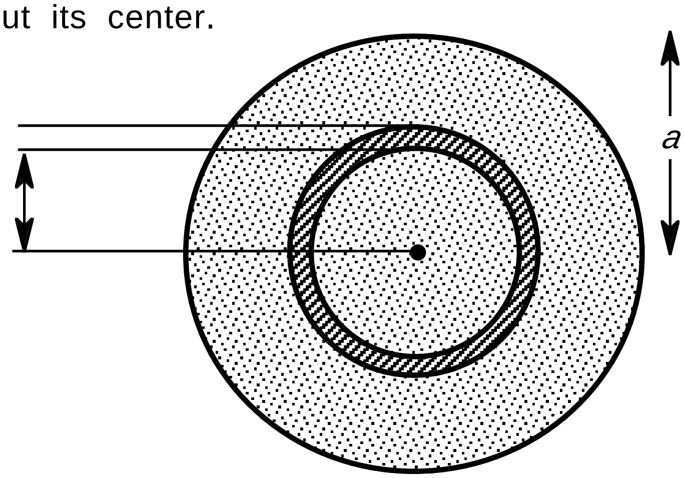
$$\text{area} = \int_0^{\pi/2} \sqrt{1-(\sin\theta)^2}\cos\theta\mathrm{d}\theta = \int_0^{\pi/2} (\cos\theta)^2\mathrm{d}\theta = \int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\mathrm{d}\theta$$

$$= \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi/2} = \frac{\pi}{4}.$$

Thus area=$\pi/4$, as expected from geometry.

Example: In mechanics, the moment of a surface about a point perpendicular to the surface is defined as $I=\int\rho^2 dA/A$ over the surface where ρ is the distance from the point to an infinitesimal area of the surface dA and A is the total area of the surface. Find the moment of a disk about its center.

Solution:



Break the circle up into a series of rings of infinitesimal thickness dρ and inner radii ρ. The area of each ring is then

$$dA=\pi(\rho+d\rho)^2-\pi\rho^2=2\pi\rho d\rho.$$

Since the area of the circle is πa^2, the moment of the circle about its center is

$$I=\frac{\int_0^a \rho^2 dA}{\pi a^2}=\frac{2\pi\int_0^a \rho^3 d\rho}{\pi a^2}=\frac{a^2}{2}.$$

DIFFERENTIAL EQUATIONS

A differential equation is just an equation involving derivatives. Lots of differential equations pop up in all areas of physics. Solving a differential equation is, in general, tricky work because there is no simple algorithm which works every time. Here, however, is an interesting example in which solution is straightforward.

Example: The compound interest law states that the time rate of change in the magnitude of a quantity is proportional to the magnitude of the quantity. It can be written as the differential equation

$$dx/dt = kx.$$

It can apply to a great many physical situations; for example x can represent the population of viruses in a petri dish, the amount of a drug in the body, or the mass of radioactive material. Solve this differential equation.

Solution: The solution here is simple. Just factor all the stuff with x's in it to one side of the equation and the stuff with t's to the other like this, $dx/x = k\,dt$. Now integrate both sides,

$$\int dx/x = \ln x + \ln C = \int k\,dt = kt$$

where C and hence $\ln C$ is some as yet undetermined constant. Combining terms and taking the anti-logarithm, this becomes

$$Cx = e^{kt}.$$

If x_o is the magnitude of x when $t=0$,

$$x = x_o e^{kt}.$$

This is the famous law of exponential decay (if $k<0$) or exponential increase (if $k>0$). Fixing the magnitude of the constant C is an example of applying a <u>boundary value</u> to the variables.

This equation is soluble because the variables are separable, i.e. it's possible to rearrange the equation to move the x stuff to the left and the t stuff to the right. This is not usually the case.

APPROXIMATIONS, NUMERICAL METHODS and COMPUTER ALGEBRA SYSTEMS

Since scientists are more interested in the answer than in mathematical elegance, they often use various brute force methods for dealing with differential equations. One way is to approximate the differentials in physical laws with finite increments. For example, we might write the compound interest law as $\Delta x = kx\Delta t$, a form adequate to calculate changes in Δx when there are small changes in time Δt.[3]

Numerical methods can be used to evaluate integrals and differential equations. We showed a simple method in the section on **INTEGRAL CALCULUS** and a host of more elaborate techniques have evolved.

But the coolest way to do calculus nowadays is with computer algebra systems like Maple™, Mathematica™, or Derive™ which run on a microcomputer. These programs can be told to evaluate a differential or integral, or solve a differential equation, and will come back with a solution in closed form, if possible, or a numerical solution otherwise. Unfortunately the programs are **not** especially friendly and require considerable training—and considerable knowledge of calculus—to use.

[1]A mathematician would, quite properly, cringe at the idea of “dividing” two differentials to produce a derivative but in this discussion we'll stick to rough and ready intuitive approaches.
[2]Mathematicians have some pretty elaborate definitions of “reasonable functions” which we'll skip here.
[3]But what is a “small” change in time? That really depends on the degree of accuracy required and varies with the problem. Let's not think about it any more.