TRIGONOMETRY

BASICS

Trigonometry deals with the relation between the magnitudes of the sides and angles of a triangle. In a typical trigonometry problem, two angles and a side of a triangle would be given and the length of the other two sides calculated. Trigonometry grew from the study of practical problems in surveying and has obvious application there. But trigonometry is also indispensable to the study of almost every area of physics including mechanics, optics, acoustics, electromagnetism, and quantum mechanics.

ANGLES

In trigonometry angles are measured from the positive x axis of the Cartesian coordinate system, as shown in figure 1. Angles measured counterclockwise, as θ is in figure 1, are positive, and angles measured clockwise are negative. Angles can be measured in degrees (°) or radians (rad). The conversion from degrees to radians is $180° = \pi$ rad $= 3.14159...$ rad.

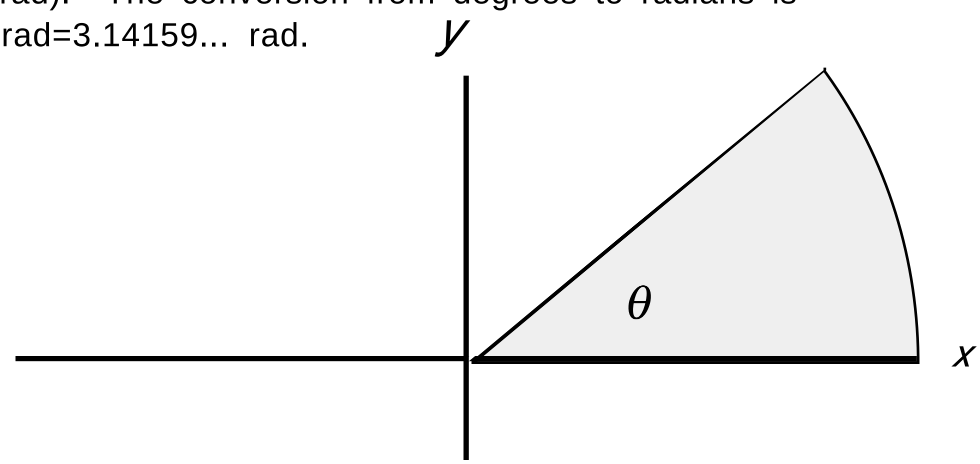


Figure 1. Trigonometric measurement of angle θ.

To get a feeling for the units, note that a right angle is 90° or $\pi/2$ rad. Figure 2 shows some angles in both units

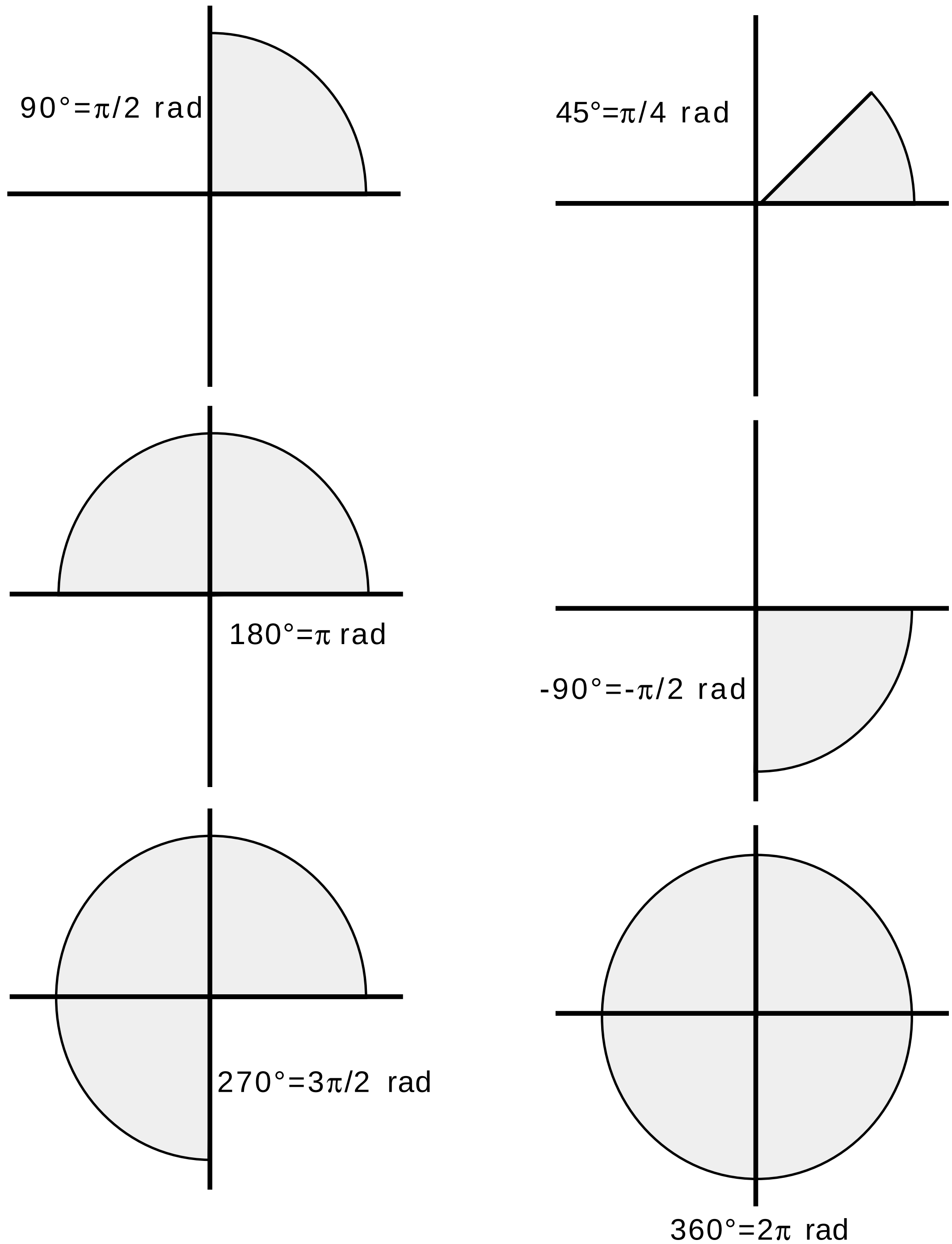


Figure 2. Some angles in degrees and radians.

TRIGONOMETRIC FUNCTIONS

Crucial to the study of trigonometry are six trigonometric functions of an angle. For an angle θ, these are the ratios of the various sides of a right triangle on θ. Figure 3 shows such a triangle with its sides *a, b*, and *c*. Side *c* opposite the right angle is called the hypotenuse, while the other two sides are called the legs of the triangle.

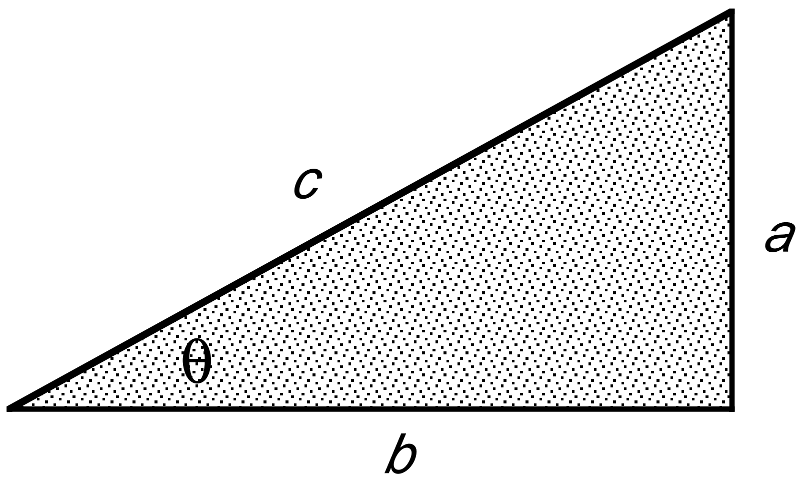


Figure 3. A right triangle of sides *a*, *b*, and hypotenuse *c* with angle θ .

With reference to the diagram above, the six trigonometric functions and their abbreviations are:

$$\text{sine}\theta = \sin\theta = a/c$$
$$\text{cosine}\theta = \cos\theta = b/c$$
$$\text{tangent}\theta = \tan\theta = a/b$$
$$\text{cotangent}\theta = \cot\theta = b/a$$
$$\text{secant}\theta = \sec\theta = c/b$$
$$\text{cosecant}\theta = \csc\theta = c/a$$

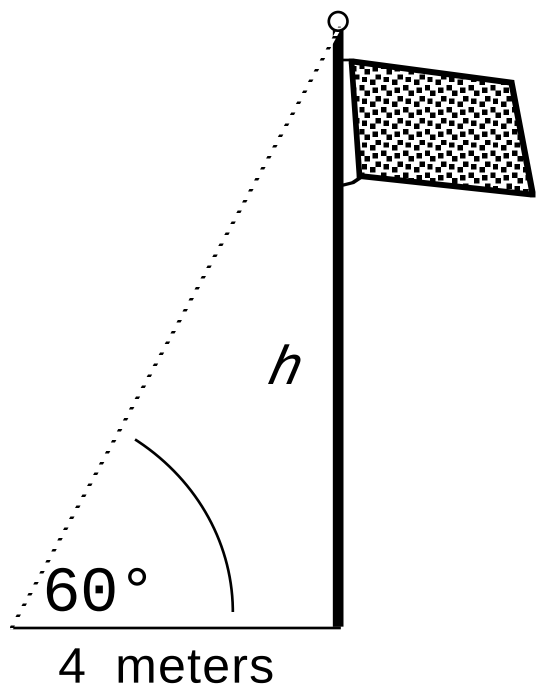
In olden times, values of the trigonometric functions were compiled in books of tables somewhat like that of figure 4. Nowadays electronic calculators do the work.

angle	sin	tan	cot	cos
0°	0.000	0.000	∞	1.000
30°	0.500	0.577	1.732	0.866
45°	0.707	1.000	1.000	0.707
60°	0.866	1.732	0.577	0.500
90°	1.000	∞	0.000	0.000

Figure 4. A much abbreviated trigonometric table.

Here's an example so you can see what to do with all this.

Example: Four meters from the base of a flagpole the top of the flagpole subtends an angle of 60° at ground level. How tall is the flagpole?



From the diagram and the definition of the tangent, height h satisfies $h/(4\text{m}) = \tan 60°$ so $h = (4\text{m})\tan 60°$. From the table of figure 4 or from a calculator, $\tan 60° = 1.732$ so the height of the flagpole is $h = 4 \times 1.732 = 6.928$ meters.

RANGE OF TRIGONOMETRIC FUNCTIONS

Trigonometric functions cannot, in general, take on just any value. For example, in figure 3, $a \leq c$, so $|\sin\theta| = a/c \leq 1$. Likewise $|\cos\theta| = b/c \leq 1$. The tangent, on the other hand is unbounded and can take on any value.

SIGN CONVENTIONS

In a right triangle like that of figure 3, $\theta \leq 90°$ and all the trigonometric functions of θ are positive. We can extend the definition of trigonometric functions by considering triangles drawn in a Cartesian coordinate system as in figure 5.

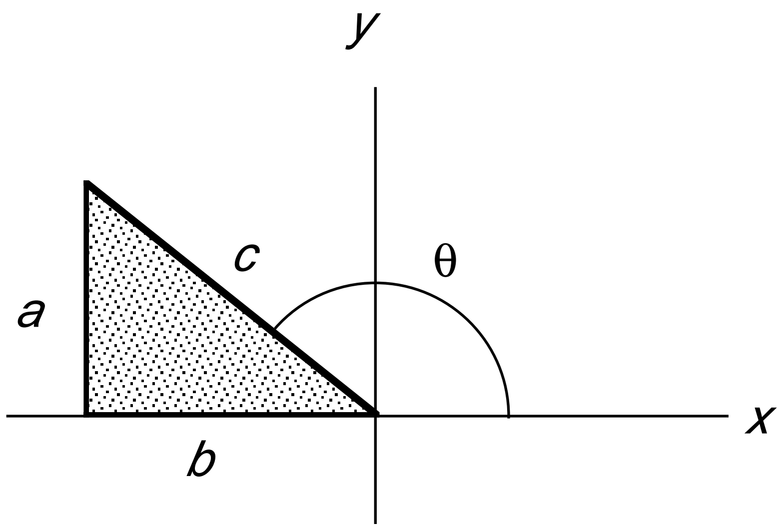


Figure 5. A right triangle in Cartesian coordinates

The trigonometric functions of angle θ are defined as

$$\sin\theta = a/c$$
$$\cos\theta = b/c$$
$$\tan\theta = a/b$$

Since c and a are positive and b is negative, the sine is positive and cosine and tangent are negative. The actual numerical value of the functions can be related to that of an angle in the first quadrant by noting that

$$\sin(\theta) = \sin(180°-\theta)$$
$$\cos(\theta) = -\cos(180°-\theta)$$
$$\tan(\theta) = -\tan(180°-\theta)$$

Hence $\sin(120°)=\sin(180°-120°)=\sin 60°=0.866$.

The sign of the trigonometric functions depends only on the quadrant in which angle θ lies. Figure 6 gives the range of the sine, cosine, and tangent in each of the four quadrants.

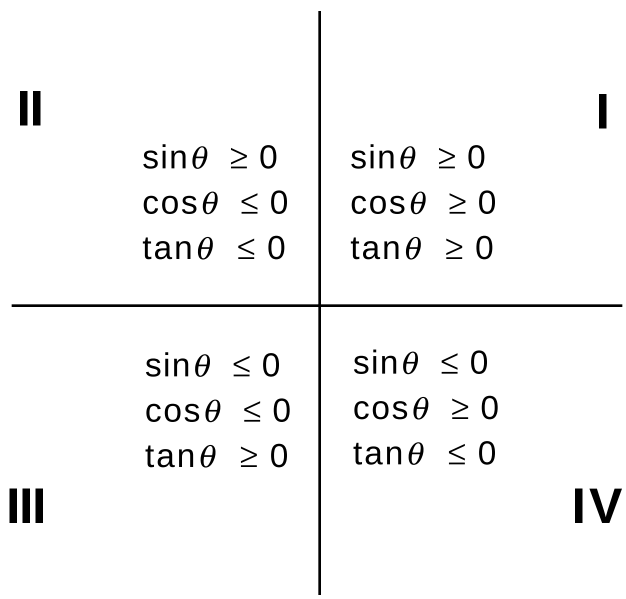


Figure 6. Signs of the trigonometric functions in each quadrant of the Cartesian coordinate system.

TRIGONOMETRIC IDENTITIES

A number of algebraic relationships or *identities* exist among the trigonometric functions. Some of the more useful ones are tabulated below.

$$
\begin{aligned}
\sin(-\theta) &= -\sin\theta \\
\cos(-\theta) &= \cos\theta \\
\sin(90°\pm\theta) &= \cos\theta \\
\cos(90°-\theta) &= +\sin\theta \\
\cos(90°+\theta) &= -\sin\theta \\
\sin(180°+\theta) &= -\sin\theta \\
\sin(180°-\theta) &= +\sin\theta \\
\cos(180°\pm\theta) &= -\cos\theta \\
\sin\theta &= 1/\csc\theta \\
\cos\theta &= 1/\sec\theta \\
\tan\theta &= 1/\cot\theta = \sin\theta/\cos\theta \\
\sin^2\theta + \cos^2\theta &= 1 \\
\sin 2\theta &= 2\sin\theta\cos\theta \\
\cos 2\theta &= 2\cos^2\theta-1 = 1-2\sin^2\theta = \cos^2\theta-\sin^2\theta \\
\sin\theta\sin\phi &= [\cos(\theta-\phi)-\cos(\theta+\phi)]/2 \\
\cos\theta\cos\phi &= [\cos(\theta-\phi)+\cos(\theta+\phi)]/2 \\
\sin\theta\cos\phi &= [\sin(\theta-\phi)+\sin(\theta+\phi)]/2
\end{aligned}
$$

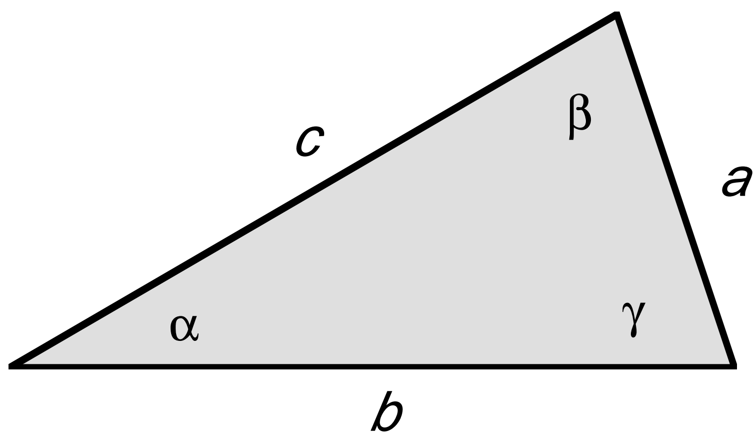
Table 1. Some useful trigonometric identities.

It's easy to demonstrate the validity of these identities by substituting in actual angles. Taking $\theta=30°$, for example,

$$\sin^2\theta+\cos^2\theta = (0.500)^2+(.866)^2=1.00,$$

thus verifying the 12th identity above. Actually proving an identity requires some algebraic and geometric effort. For example, from figure 3 and the Pythagorean theorem, $a^2+b^2=c^2$. Dividing both sides of this latter equation by c^2 and using the definitions of sine and cosine we derive the identity we just verified.

LAW OF COSINES and LAW OF SINES



The law of sines and the law of cosines are two useful theorems which apply to **any** triangle like the one above, not just right triangles.

The law of sines states that the ratio of the sine of an angle to the length of the opposite side is the same for any angle of a triangle. In equation form when applied to the diagram above this becomes

$$\sin\alpha/a=\sin\beta/b=\sin\gamma/c.$$

The law of cosines applied to the diagram above is

$$c^2=a^2+b^2-2ab\cos\gamma.$$

Note that when $\gamma=90°$, the law of cosines reduces to the Pythagorean theorem.

PLOTTING TRIGONOMETRIC FUNCTIONS

Plotting the sine and cosine as a function of angle gives a sinusoidal curve like the one shown in figures 7 and 8.

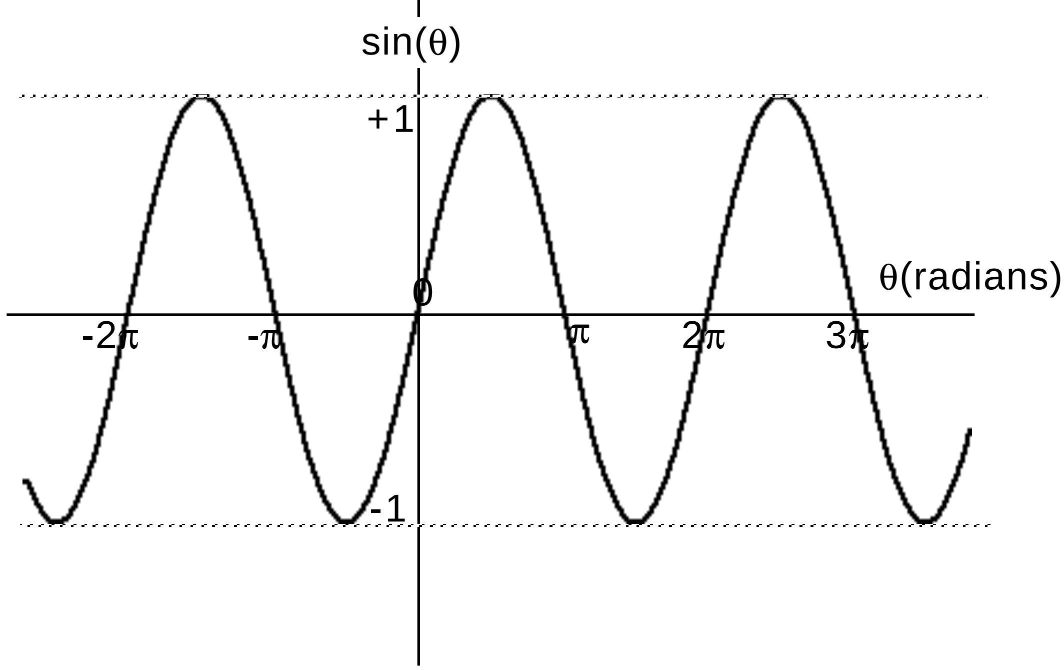


Figure 7. The sine plotted as a function of angle.

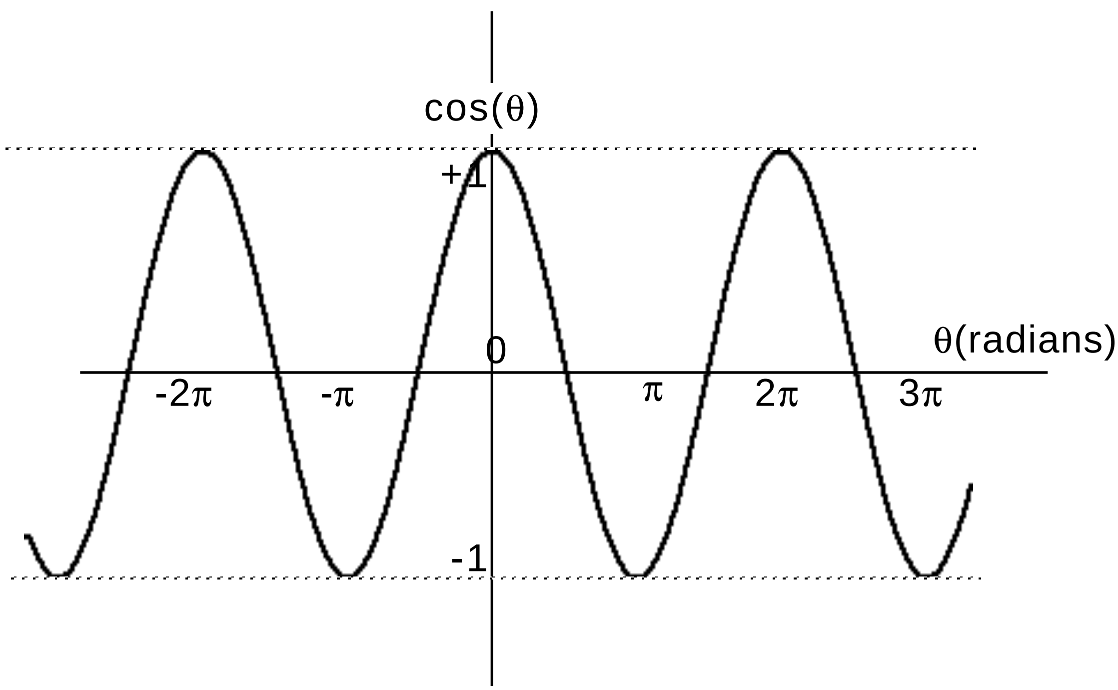


Figure 8. The cosine plotted as a function of angle.

Note that the sine and cosine curves have the same shape but are shifted with respect to one another by an angle $\pi/2=90°$.

SMALL ANGLE APPROXIMATIONS

For small angles, $\sin\theta \cong \theta \cong \tan\theta$, and $\cos\theta \cong 1$ if θ is given in radians. This is demonstrated by the plot of figure 9. **Remember, this approximation only works when the angle is given in radians.**

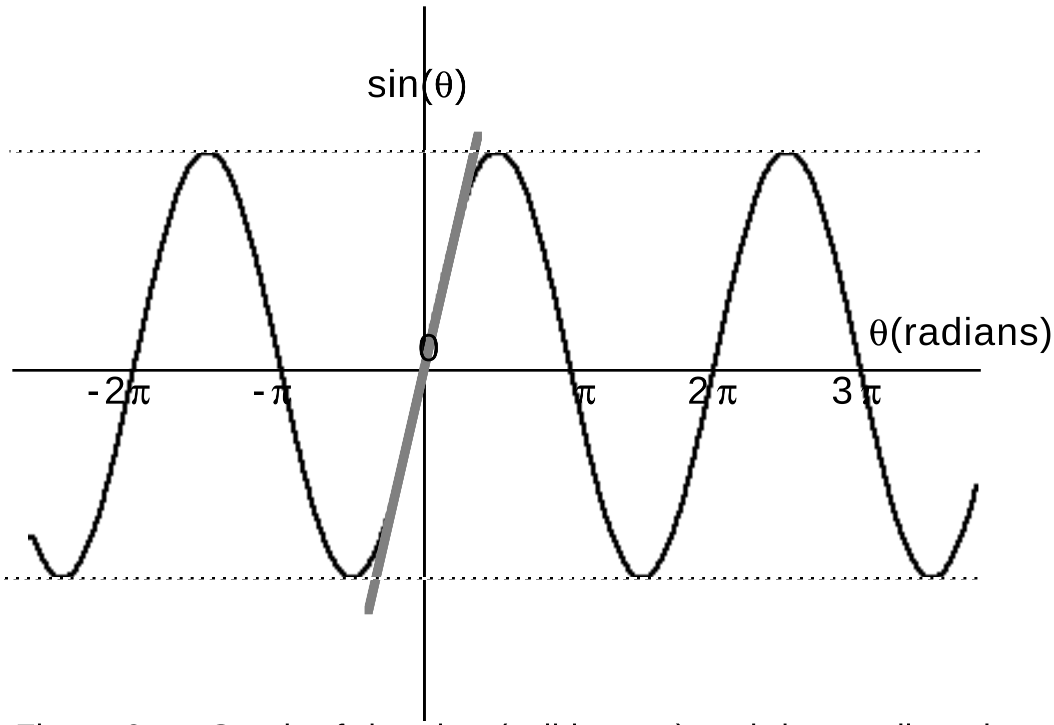


Figure 9. Graph of the sine (solid curve) and the small angle approximation to the sine (gray curve).

INVERSE TRIGONOMETRIC FUNCTIONS

Suppose the cosine of an angle takes on a value x. Then θ is given by $\theta=\text{arc cos}\,x$, where "arc cos" is the arc cosine or inverse cosine of x.. Sometimes arc $\cos x$ is written $\cos^{-1}x$. Arc sine, arc cosine, and arc tangent can be obtained from working backward from tables like figure 4 or with a calculator.

Do not confuse an inverse function with the inverse of a function. Remember that in general $\sin^{-1}\theta \neq (\sin\theta)^{-1} = 1/\sin\theta$.

One peculiarity of inverse trigonometric functions is that they are multiple valued, that is there are many (actually, an infinite number) of solutions to equations like $\theta = \arcsin x$. For example, using a calculator, we find $\sin 160° = 0.342\ldots$ But again using the calculator we find that $\arcsin(0.342\ldots) = 20°$! Checking again we find that $\sin 20° = 0.342\ldots$, so the calculator is functioning fine and 20° *is*, in fact, a solution to our equation. So would -200°, -340°, 380°, and 520° since the sine of any of these angles is 0.342… So why isn't $\sin^{-1}(\sin\theta) = \theta$? The answer in this case goes back to the identity $\sin(180° - \theta) = \sin\theta$, where $\theta = 20°$ here. We can generate still more solutions from the identity $\sin\theta = \sin(\theta \pm 360°)$. The calculator only returns one value of a trigonometric function, the <u>principle value</u> from which all other values may be deduced. So how do you know which value of an inverse trigonometric function is correct? In practice, that's determined by the physics of the problem in question.