

VECTORS

BASIC CONCEPTS

Scalar quantities are just numbers like 10, 3.6×10^{-8} , π . Examples of physical quantities which are scalar variables are temperature, mass, density, voltage, electrical charge, electrical current, energy, power. Variables representing scalar quantities are written as ordinary letters like x , y , θ .

Vectors are mathematical entities which have a magnitude and direction. Examples of physical quantities which are vectors are direction, velocity, acceleration, force, torque, electrical field. Variables representing vector quantities are usually written in boldface like \mathbf{X} , \mathbf{j} , \mathbf{N} .

A vector quantity is represented geometrically by an arrow in the appropriate direction with a length proportional to the magnitude of the quantity. Here's a vector \mathbf{A} :

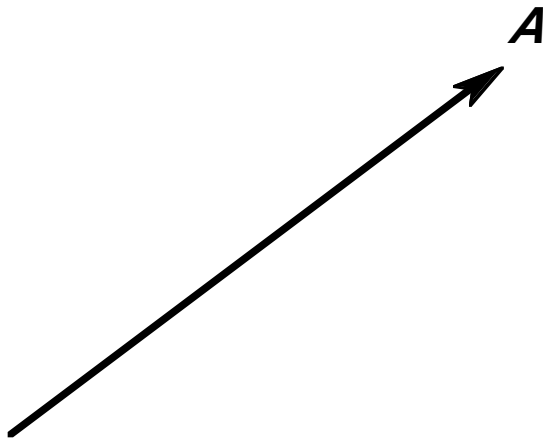


Figure 1. A vector \mathbf{A} .

A vector is not tied down to a particular location. As long as its length and direction are not altered, a vector may be placed anywhere.

It takes two quantities to specify a vector. One way is to give its magnitude and direction. Figure 2 shows the vector of figure 1 again, but this time labelled by its magnitude A and direction designated by angle θ .

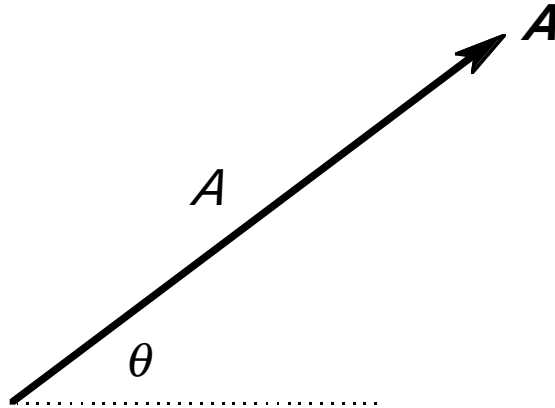


Figure 2. A vector A may be represented by its magnitude A and direction, here designated by angle θ .

For example, a vector r representing distance might be given as 14 km northeast, or 14 km 45° north of east.

Another way of designating a vector is by its components. The x and y components of a vector \mathbf{A} are symbolized by A_x and A_y . Note that A_x and A_y are scalar quantities. We can write \mathbf{A} as an ordered pair in the form $\mathbf{A} = (A_x, A_y)$. Figure 3 shows the relation between \mathbf{A} , A_x and A_y in a diagram.

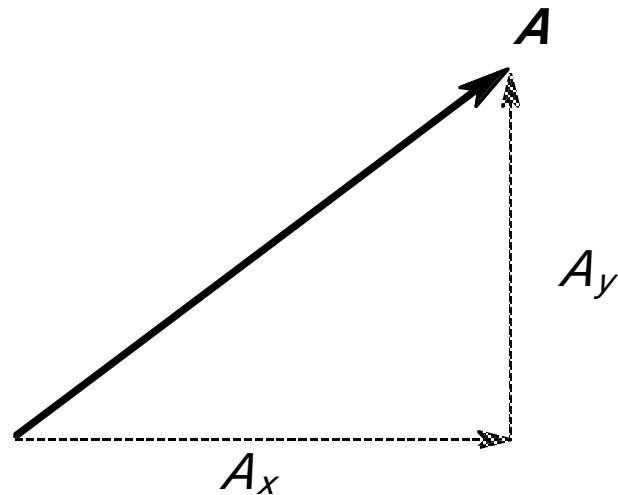


figure 3. A vector \mathbf{A} and its components A_x and A_y .

For example, the distance r above could be given as 10 km east and then 10 km north.

Very often, problems make it necessary to find A_x and A_y knowing A and θ or vice versa. With a little geometry and trigonometry, this is straightforward.

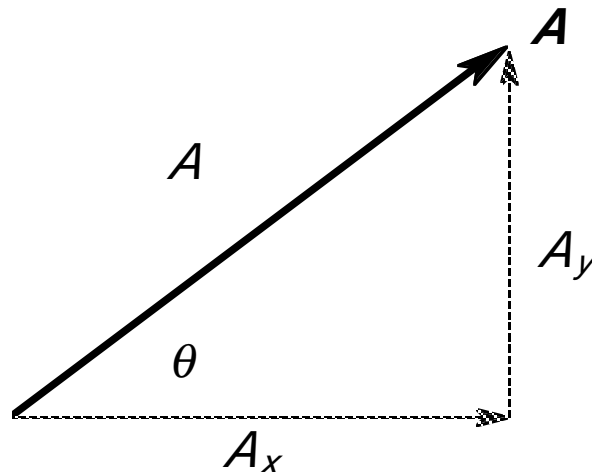


figure 4. Geometry for relating magnitude and direction to vector components.

Figure 4 combines figure 2 and figure 3. From the Pythagorean theorem,

$$A = \sqrt{A_x^2 + A_y^2} \tag{1}$$

From the definition of the tangent,

$$\tan\theta = A_y / A_x,$$

or

$$\theta = \arctan(A_y / A_x). \tag{2}$$

From the definition of the cosine,

$$\cos\theta = A_x / A,$$

or

$$A_x = A \cos\theta. \tag{3}$$

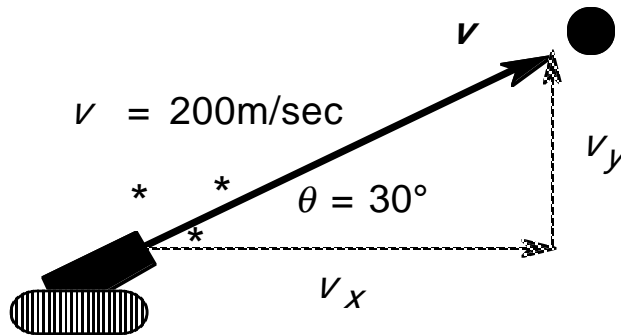
Likewise, from the definition of the sine,

$$A_y = A \sin \theta. \tag{4}$$

Equations (1)-(4) can be used for converting between representations of a vector \mathbf{A} .

Example: A cannon aimed at an angle of 30° fires a cannon ball with muzzle velocity of 200m/sec. When the cannon ball leaves the muzzle, how fast is it rising and what is the speed of its shadow moving on the ground?

Solution: First draw the vector below representing the cannon ball's velocity when it leaves the cannon:



Assuming the cannon is on a plane surface, the speed of the shadow is the horizontal component of muzzle velocity which is, from equation (3),

$$v_x = v \cos \theta = (200\text{m/sec})\cos(30^\circ) = 173\text{m/sec}.$$

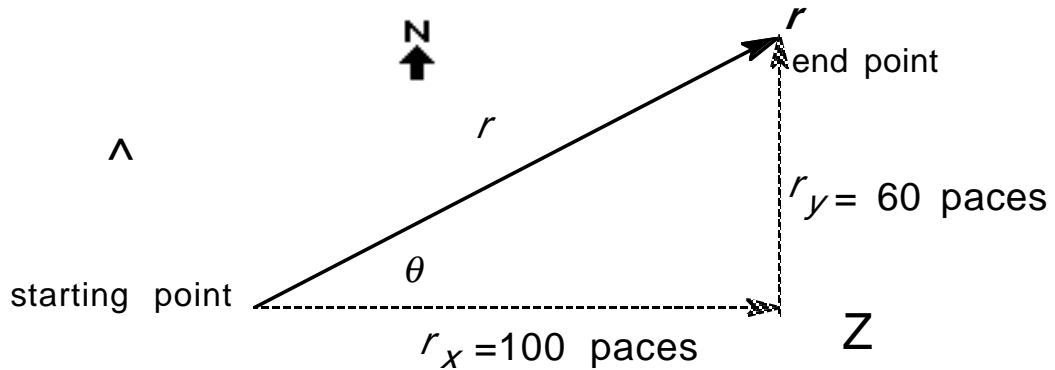
(Fun fact: from the laws of physics the speed of the shadow will be the same throughout the cannon ball's flight.)

Likewise, from equation (4), the vertical component of velocity which is the speed at which the cannon ball rises is

$$v_y = v \sin \theta = (200\text{m/sec})\sin(30^\circ) = 100\text{m/sec}.$$

Example: A boy scout running a compass course is told to go 100 paces east and 60 paces north. How many paces in what direction should he walk to reach his destination in an efficient straight line?

Solution: First draw off the compass course in the diagram below:



In this problem, the position vector r which extends from the starting point to the end point of the compass course is, in component form, $r=(r_x, r_y)$. From (1), the magnitude of the vector is

$$r=\sqrt{(r_x^2+r_y^2)}=\sqrt{[(100 \text{ paces})^2+(60 \text{ paces})^2]}=117 \text{ paces.}$$

From equation (2), the angle θ in the diagram is

$$\theta=\text{arc tan}(r_y/r_x)=\text{arc tan}(60 \text{ paces}/100 \text{ paces})=\text{arc tan}(0.6)=31^\circ.$$

The boy scout can reach his end point by walking 117 paces at 31° north of east thus saving $100+60-117=43$ paces!

VECTOR ADDITION

Vectors may be added graphically or trigonometrically. The graphic method is more appealing intuitively, the trigonometric method more accurate. Both give the same result.

Suppose we want to add two vectors A and B shown in figure 1.

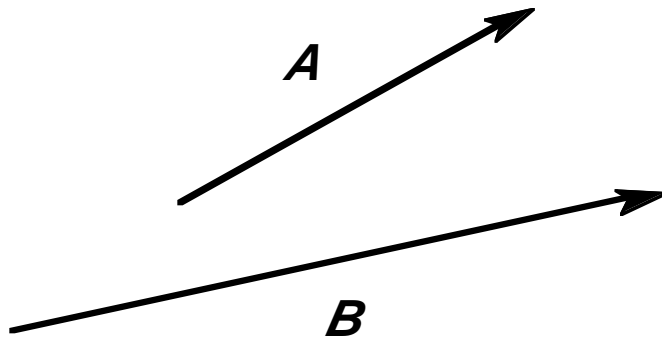


Figure 1. Vectors A and B .

There are two different, but equivalent graphical methods. The first is the parallelogram method shown in figure 2.

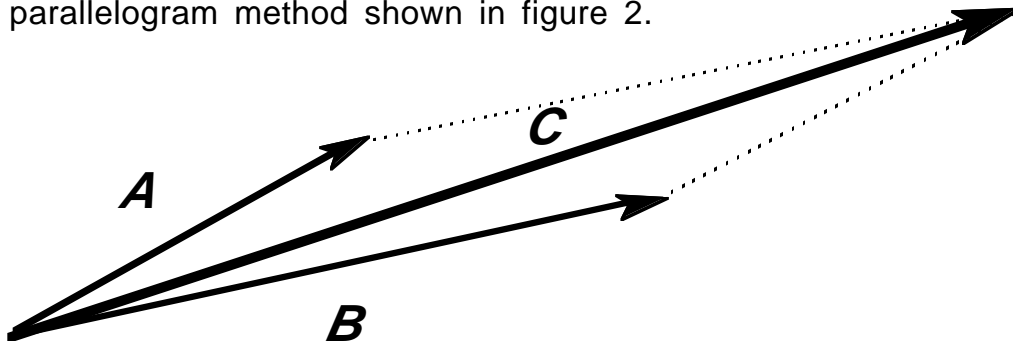


Figure 2. Parallelogram method of constructing $C = A + B$.

In the parallelogram method, A and B are drawn with their tails coinciding. These two vectors are used as the sides of a parallelogram and a new vector is drawn as the diagonal of the parallelogram. This new vector which we'll call C is the sum of vectors A and B . In vector algebra we'd write $C = A + B$. If vectors A and B have been carefully drawn to scale, the magnitude C of vector C is proportional to its length in the

diagram. Likewise, its orientation with respect to some reference line like the north-south axis of geography or the horizontal axis of ordinary trigonometry may be measured with a protractor.

The second way is to redraw the two vectors such that the head of vector A touches the tail of vector B , as shown in figure 2.

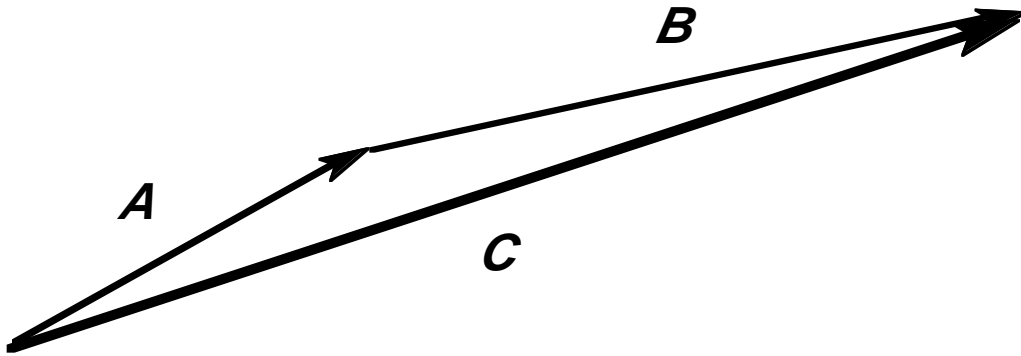


Figure 3. Head to tail addition of vectors A and B to get vector C .

Now draw a new vector from the tail of A to the head of B . This vector, C , is the sum of A and B . From geometry it is clearly the same as the vector C in figure 2.

The head-to-tail method requires less drawing than the parallelogram method and additionally, it is easier to generalize to summing multiple vectors. Figure 4, for example shows how to form the sum $U+V+W+X+Y=Z$.

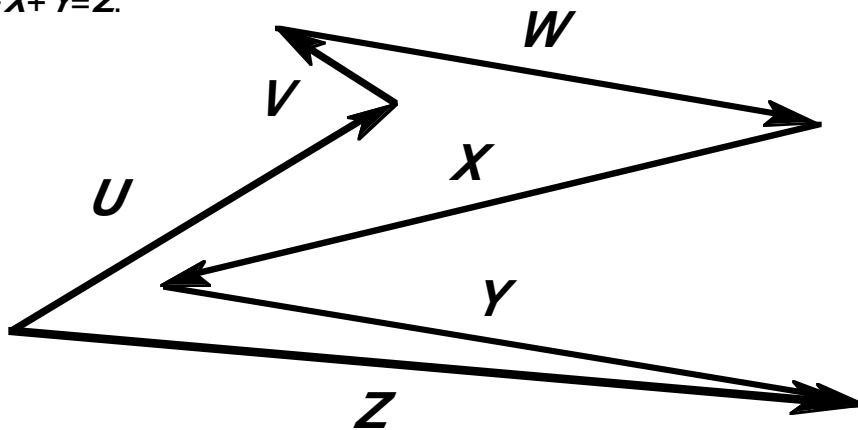


Figure 4. Head-to-tail formation of the vector sum $Z=U+V+W+X+Y$.

Scaling out diagrams is tedious business and its accuracy is limited. Usually vectors are added using an analytic method. If two vectors are given in component form, adding them is very easy. Consider figure 5 in which vectors $\mathbf{A}=(A_x, A_y)$ and $\mathbf{B}=(B_x, B_y)$ are added to form vector $\mathbf{C}=(C_x, C_y)$.

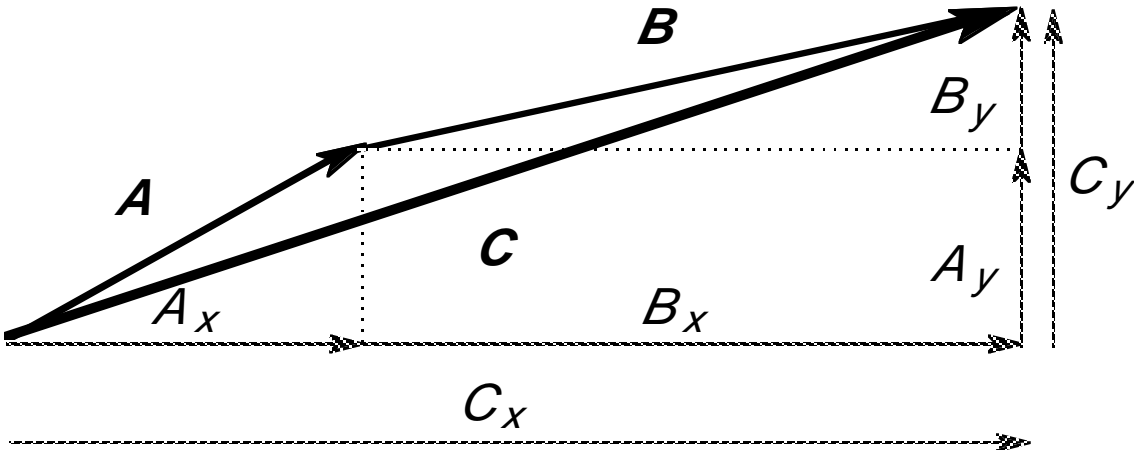
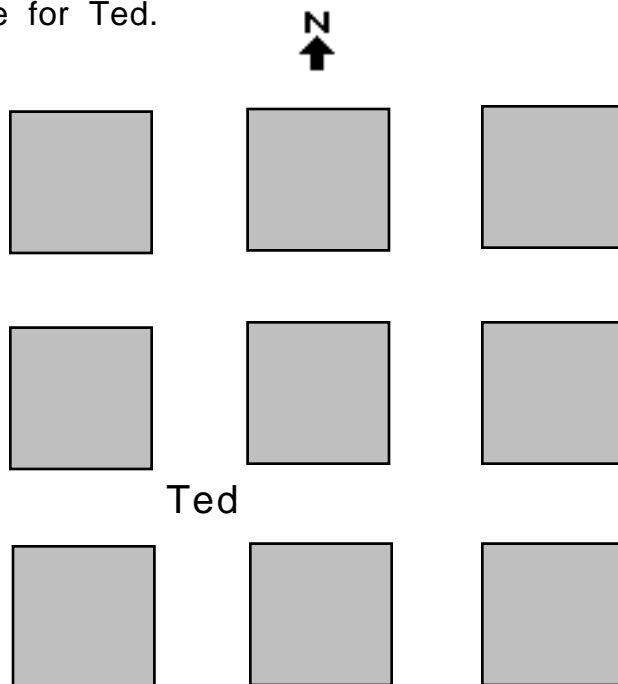


Figure 5. Adding vectors \mathbf{A} and \mathbf{B} by components.

From figure 5 it is clear that $C_x=A_x+B_x$ and $C_y=A_y+B_y$. In ordered pair notation we would write

$$\mathbf{C}=(C_x, C_y)=\mathbf{A}+\mathbf{B}=(A_x, A_y)+(B_x, B_y)=(A_x+B_x, A_y+B_y) \quad (1)$$

Example: Bill tells Ted that to get to the pizza parlor, Ted should go 5 blocks east and 4 blocks north to the video arcade and then 2 blocks west and 1 block north to the pizza parlor. Use vector addition to find a more efficient route for Ted.



Solution: The diagram above shows Ted and a few blocks of the city. In ordered pair notation, the distance from the starting point to the video arcade is $\mathbf{A}=(5 \text{ blocks}, 4 \text{ blocks})$. The distance from the arcade to the pizza parlor is $\mathbf{B}=(-2 \text{ blocks}, 1 \text{ block})$. (The negative sign comes in since the distance is west.) From equation (1) the distance \mathbf{C} from the starting point to the pizza parlor is

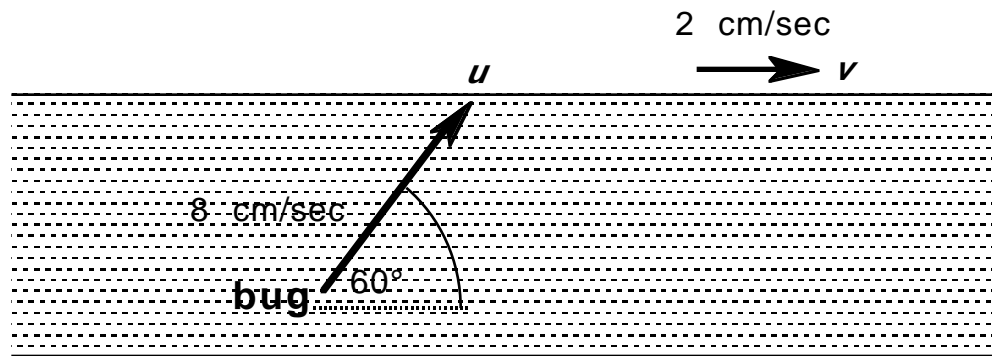
$$\mathbf{C}=\mathbf{A}+\mathbf{B}=(5 \text{ blocks}, 4 \text{ blocks})+(-2 \text{ blocks}, 1 \text{ blocks})$$

$$=(3 \text{ blocks}, 5 \text{ blocks}).$$

The most efficient route from the starting point to the pizza parlor is given by vector \mathbf{C} and is 3 blocks east and then 5 blocks north.

In most problems, magnitude and direction of vectors are given. To add two vectors given in these terms, first convert each vector to component form, add the components using equation (1), and then convert the result back to a magnitude and direction. This is illustrated in the following example.

Example: A conveyor belt moves to the right with a speed of two centimeters per second. A bug runs on the conveyor belt with a speed of eight centimeter per second relative to the conveyor belt along a line making an angle of 60° with its direction of movement. What is the bug's velocity with respect to the ground?



Solution: The diagram above shows the bug and the belt along with their velocities u and v given in the problem. Assume the belt is moving in the x direction, as the diagram shows. The total velocity of the bug is $w = u + v$. To find w , first break up u and v into components. The components of u are

$$u = ([8 \text{ cm/sec}][\cos 60^\circ], [8 \text{ cm/sec}][\sin 60^\circ]) \\ = (4 \text{ cm/sec}, 6.9 \text{ cm/sec}).$$

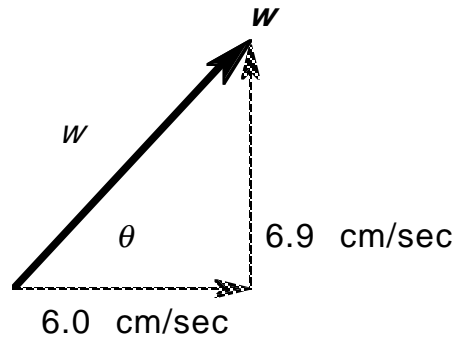
Since v is parallel to the x axis, its components are

$$v = (2 \text{ cm/sec}, 0).$$

The sum of u and v is

$$w = u + v = (4 \text{ cm/sec}, 6.9 \text{ cm/sec}) + (2 \text{ cm/sec}, 0) \\ = (6 \text{ cm/sec}, 6.9 \text{ cm/sec}).$$

The diagram below shows vector w .



From the diagram and the Pythagorean theorem, the magnitude of w is

$$w = \sqrt{[(6.0 \text{ cm/sec})^2 + (6.9 \text{ cm/sec})^2]} = 9.1 \text{ cm/sec.}$$

From the diagram and the definition of the tangent,

$$\theta = \text{arc tan}[(6.9 \text{ cm/sec}) / (6.0 \text{ cm/sec})] = \text{arc tan}(1.15) = 49^\circ.$$

So with respect to the ground, the bug is moving at 9.1 cm/sec at an angle of 49° to the direction of travel of the conveyor belt.

UNIT VECTORS

A unit vector is a vector that has unit magnitude or length. So if \mathbf{u} is a unit vector, then $|\mathbf{u}|=1$.

Some useful unit vectors are those parallel to the axes of a Cartesian coordinate system. Customarily, the unit vectors \mathbf{i} , \mathbf{j} , and \mathbf{k} are parallel to the x , y , and z axes, respectively. In component form these may be written

$$\mathbf{i}=(1,0,0), \mathbf{j}=(0,1,0), \mathbf{k}=(0,0,1).$$

Using unit vectors, a vector $\mathbf{r}=(x, y, z)$ may be written

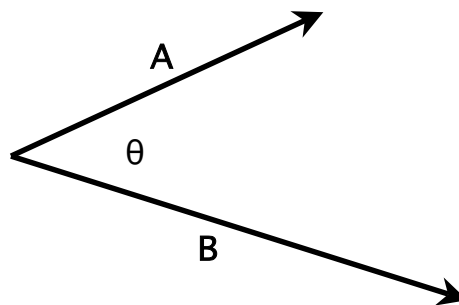
$$\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}.$$

It turns out that expressing vectors in terms of unit vectors is very helpful in the bookkeeping in vector problems.

VECTOR MULTIPLICATION

Two ways of multiplying two vectors has been defined—the dot product and the cross product.

The dot product



If two vectors \mathbf{A} and \mathbf{B} are separated by angle θ , the dot product is a scalar defined as

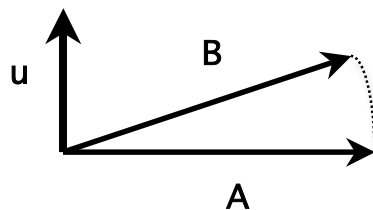
$$\mathbf{A} \cdot \mathbf{B} = AB \cos\theta.$$

In terms of Cartesian components the dot product can be written

$$\mathbf{A} \cdot \mathbf{B} = (A_x, A_y, A_z) \cdot (B_x, B_y, B_z) = A_x B_x + A_y B_y + A_z B_z.$$

The cross product

If two vectors \mathbf{A} and \mathbf{B} are separated by angle θ , the cross product is a vector defined as $\mathbf{A} \times \mathbf{B} = (AB \sin\theta)\mathbf{u}$ where \mathbf{u} is a unit vector perpendicular to the plane containing the vectors \mathbf{A} and \mathbf{B} , as shown in the diagram below.



If, using your right hand, you rotate your fingers through vector \mathbf{A} and then vector \mathbf{B} , your thumb will point in the direction of \mathbf{u} .

In Cartesian coordinates, the cross product may be written,

$$\mathbf{A} \times \mathbf{B} = \mathbf{i}(A_y B_z - A_z B_y) - \mathbf{j}(A_x B_z - A_z B_x) + \mathbf{k}(A_x B_y - A_y B_x) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

where the vertical lines $| |$ indicate a determinant.