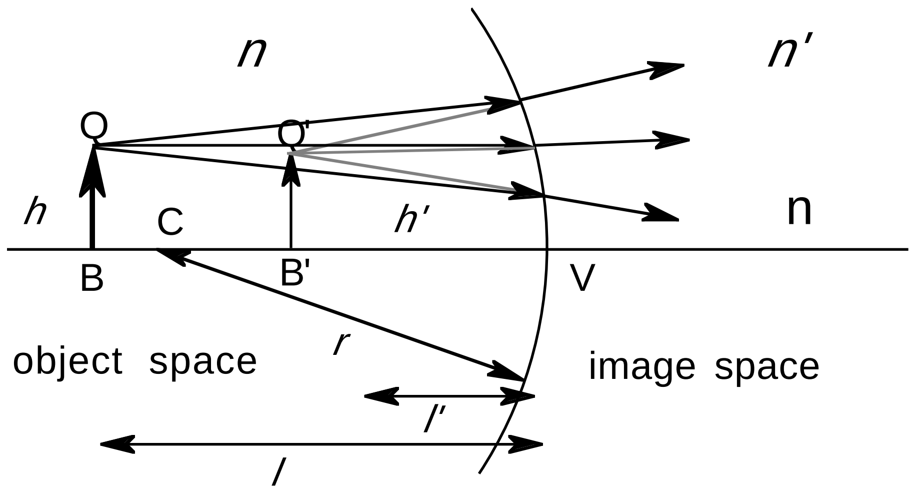
REFRACTION by SPHERICAL SURFACES

Fundamental Paraxial Equation for a Single Surface

Let two media of indices n and n' be separated by a spherical surface, as in the diagram below. What happens to the rays from object BQ and what does an observer in the second medium see when he looks at the object?



As the diagram shows, the refracted rays look to the observer as if they came from an object B'Q' instead of the actual object BQ. B'Q' is the image of BQ. It's size and position are given by the fundamental paraxial equation (we'll derive it later):

$$L' = L + F\ ,$$
$$h'/h = m = L/L'$$

where

r=VC	radius of surface curvature
l=VB	object distance
l'=V'B'	image distance
n	index of object space
n'	index of image space
$L=n/l$	reduced vergence
$L'=n'/l'$	reduced vergence
$F=(n'-n)/r$	surface power
h=BQ	object height
h'=B'Q'	image height
m	magnification

Lengths are in meters (m), vergences in diopters (D~1/m).

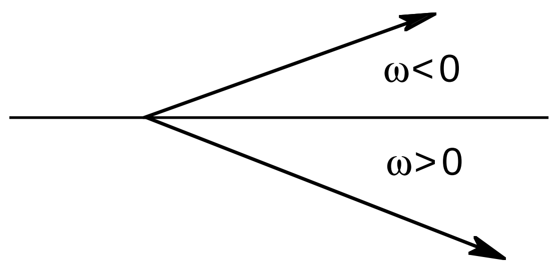
The fundamental paraxial equation follows the following sign conventions:

☞ Light travels from left to right.

☞ All horizontal distances are measured from the vertex. Distances to the right are positive, to the left negative.

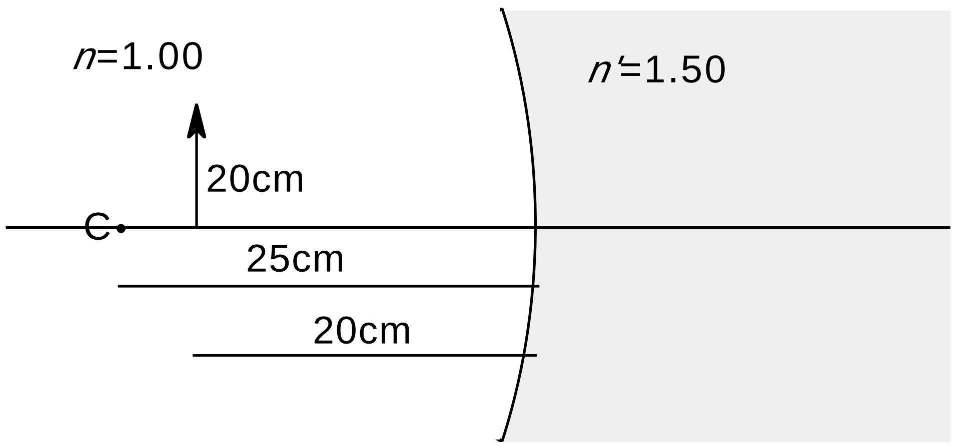
☞ All vertical distances are measured from the optical axis. Distances above the axis are positive, below negative.

☞ The angle of a ray is positive if the ray is rotated counterclockwise through the angle to reach the optic axis, negative if the ray is rotated clockwise through the angle to reach the optic axis. (See below.)



Refracting surfaces (or any other optical system) may produce real or virtual images. A real image is to the right of the vertex of the refracting surface (or the final vertex of any refracting optical system) and a virtual ∆image is to the left of the vertex of the refracting surface (or the final vertex of any refractive optical system). A real image may be projected on a screen. A virtual image cannot. Either a real or a virtual image at a point B' produces the same retinal stimulus pattern in an observer's eye.

Example: Find the location and height of the image of the object in the diagram below.



Solution: From the diagram, r=-0.25m, l=-0.20m. The surface power is given by $F=(n'-n)/r$=(1.5-1.0)/(-0.25)=-2.00D. Incoming vergence is $L=n/l$=1.00/(-0.20)=-5.00D. From the fundamental paraxial equation, the vergence of light from B when it leaves the surface is

$$L'=L+F=-5.00-2.00=-7.00\text{D}.$$

The distance to the image is

$$l'=n'/L'=1.5/(-7.00)=-0.214\text{m}=-21.4\text{cm}.$$

Object height, h, is 20cm so the height of the image is

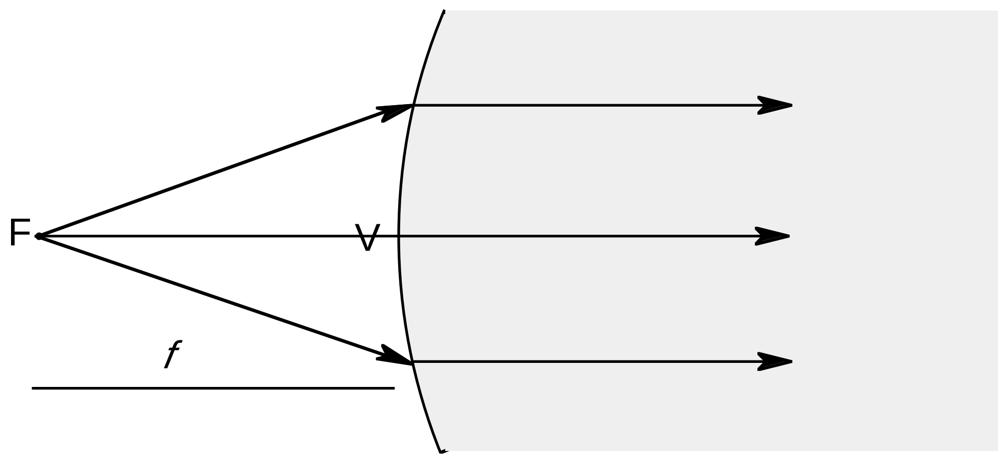
$$h'=(L/L')h=[(-5)/(-7)](20\text{cm})=+14.3\text{cm}.$$

This is a virtual, erect image.

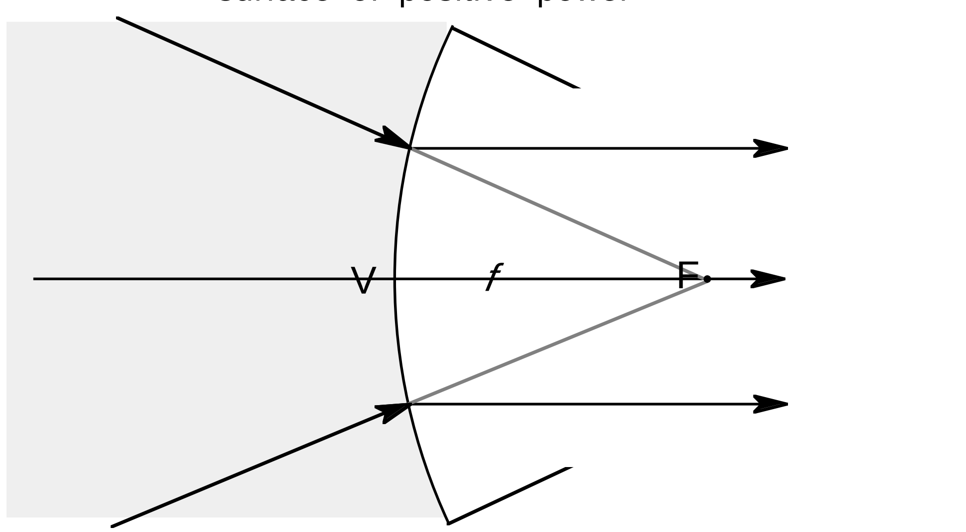
Focal Lengths

A refractive surface (or any other refracting optical system) has two focal points, a primary focal point F and a secondary focal point F'.

The primary focal point is an axial point having the property that any rays coming from it or proceding toward it travel parallel to the axis after refraction.



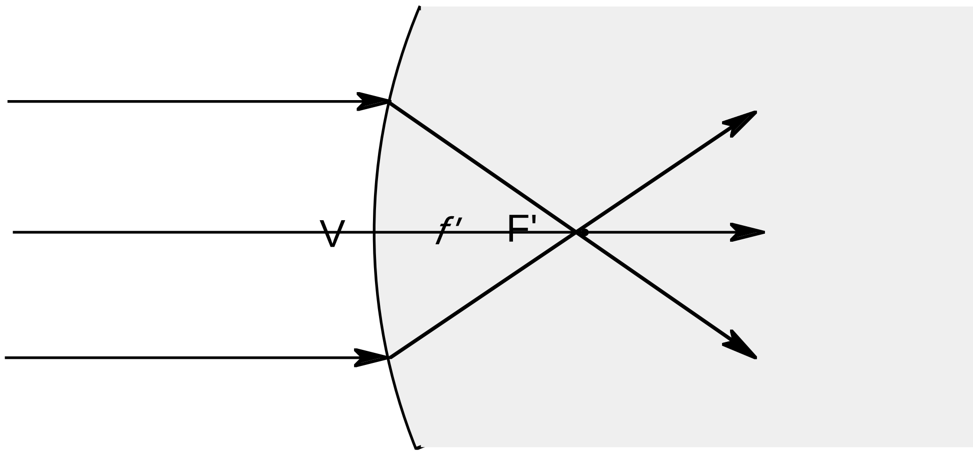
surface of positive power



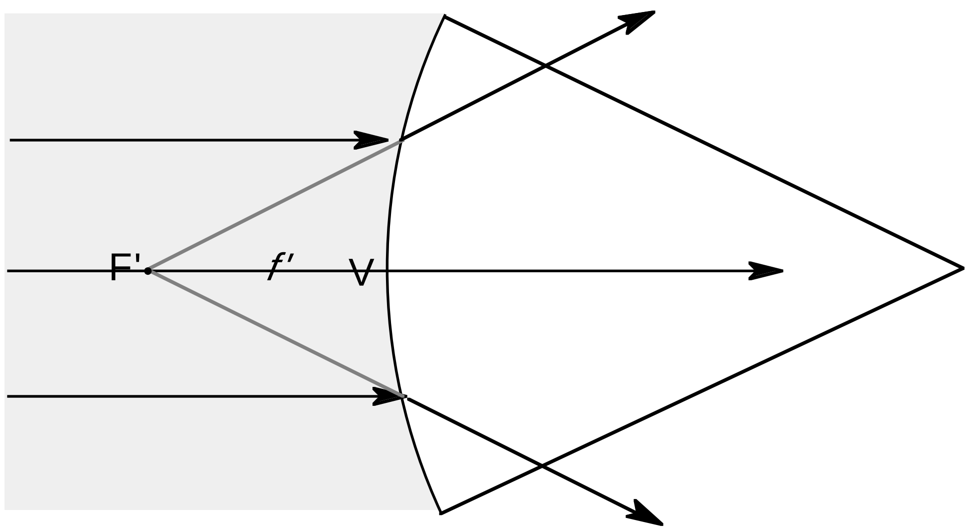
surface of negative power

The primary focal length is defined as the distance from the vertex to the primary focal point, *F*=VF, as shown in the diagrams.

The <u>secondary focal point</u>, F', is an axial point having the property that any incident ray travelling parallel to the optic axis will, after refraction, procede toward or appear to come from F'.



surface of positive power



surface of negative power

We can derive expressions for focal length in terms of surface power, F. For the primary focal length, note that an object at F will produce collimated light of zero vergence so $L'=0$. From the fundamental paraxial equation, $L'=L+F=0=n/f+F$ or, finally,

$$F=-n/f.$$

Likewise, collimated light approaching a refracting surface is zero vergence and focuses at the secondary focal point, so from the fundamental paraxial equation, $L'=L+F=n'/f'=0+F$ or, finally,

$$F=n'/f'.$$

Geometric Ray Tracing

Geometric ray tracing is useful in visualizing the solution to optical problems and verifying, at least qualitatively, what's going on.

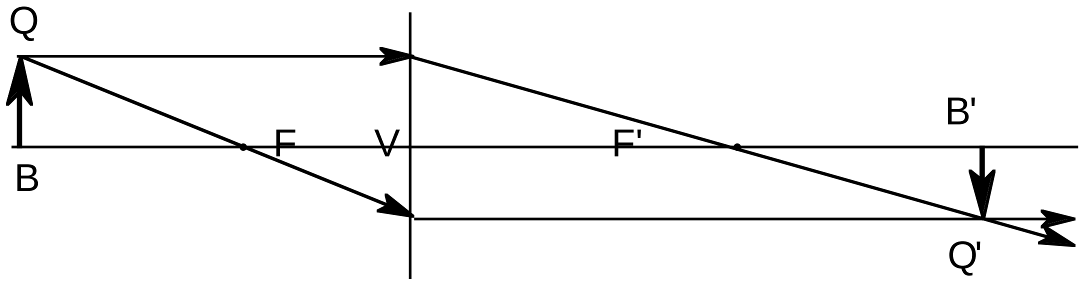
An infinite number of rays emerge from any luminous source. These rays travel in all directions. Because of the special properties of the focal points F and F' and the center of curvature, C, we may trace three of this infinitude of rays through the system. In fact any two rays would be enough to establish location and size of images. Use the following facts:

☞ Rays travelling parallel to the axis before refraction are deviated so as to pass through (or have their extensions pass through) the secondary focal point F' after refraction.

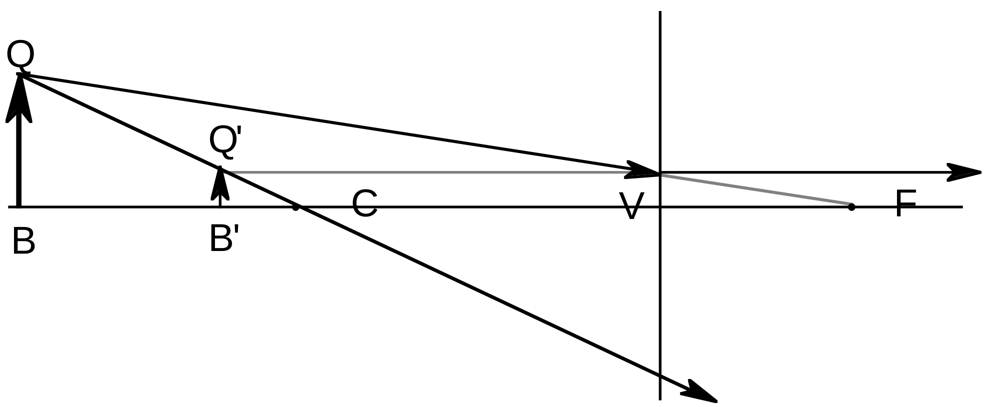
☞ Rays which intersect the primary focal point F (or have their extension intersect F) before refraction travel parallel to the axis after refraction.

☞ Rays directed toward (or from) the center of curvature of the refracting surface strike the surface normally and so are undeviated by the refracting surface.

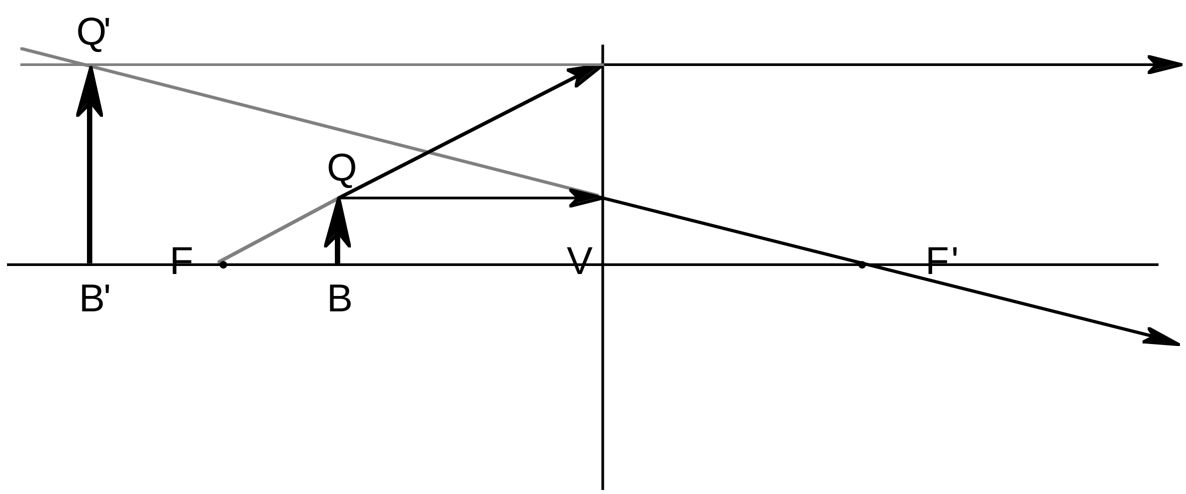
The following diagrams show some examples of using geometric ray tracing to construct the image of an object BQ. Note that the refracting surface is represented only by a straight line.



converging surface forming real inverted image



diverging surface forming virtual erect image

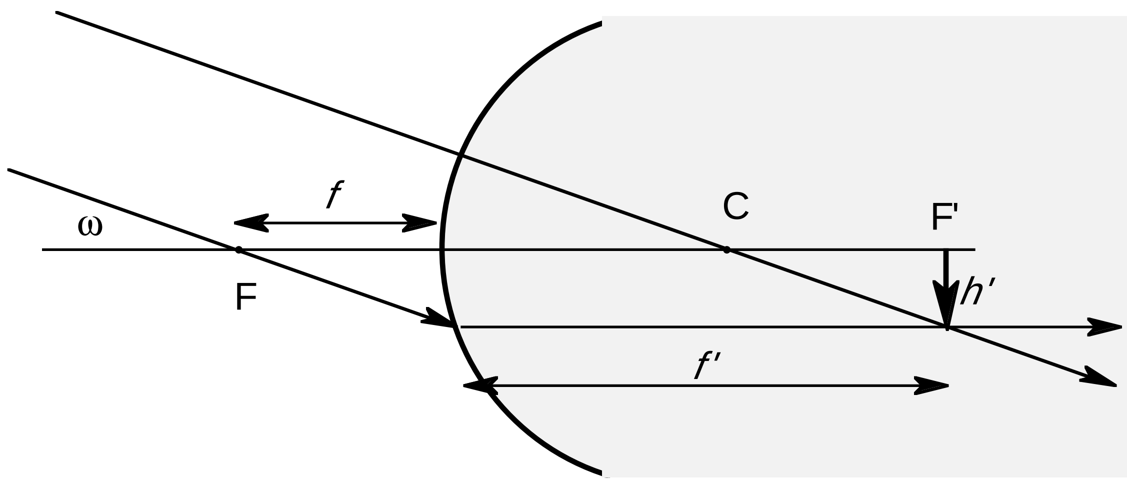


converging surface forming virtual erect image

Image of a Remote Object

Suppose a single refracting surface forms the image of a remote object, as shown in the diagram below. The image is formed at the secondary focal point of the surface and its size h' is proportional to the angle ω the object subtends at the center of curvature of the surface. From the geometry of the diagram it may be seen that

$$h' = f\omega.$$

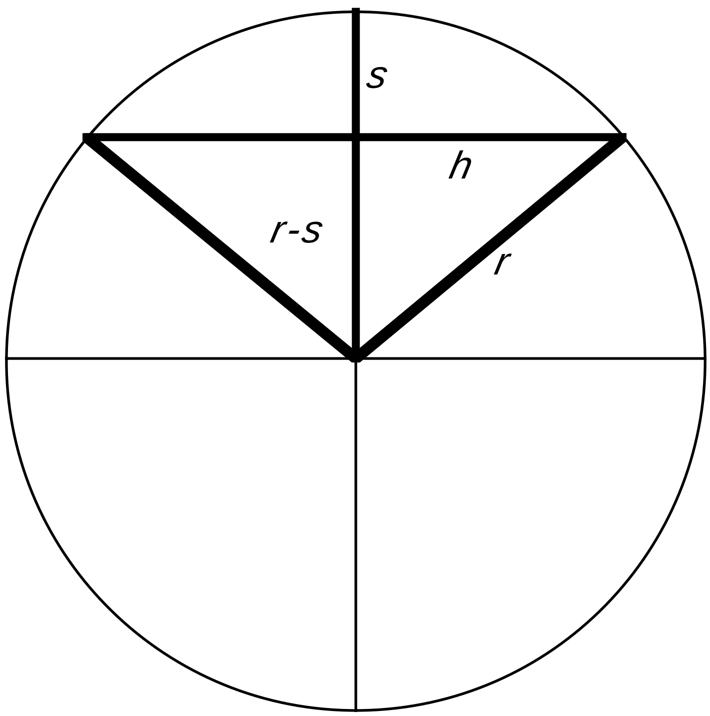


Reversibility and Conjugacy

We've noted previously that the paths of light are reversible. A consequence of this is that if an image B'Q' of height h' at B' is conjugate to an object BQ of height h at B, an object of height h' placed at B' would be conjugate to an object of height h at B.

Sagitta Formula

Consider a cross section of a sphere as shown below:



We seek a relation between the sagitta, s, the radius of curvature of the surface r, and the half diameter of the chord, h. From the Pythagorean theorem, $(r-s)^2+h^2=r^2$. Expanding terms we get

$$s^2-2rs+h^2=0.$$

Using the quadratic equation, we obtain for the sagitta

$$s=r\pm\sqrt{(r^2-h^2)}.$$

But which root should we take? Note that as $h\rightarrow 0$, $s\rightarrow 0$, so we must take the negative root. The positive root is extraneous. The final answer is, then,

$$s=r-\sqrt{(r^2-h^2)}.$$

For most practical applications (including spectacle lenses), $r^2>>h^2$ and, using the binomial theorem, the last equation reduces to the famous sagitta formula,

$$s\cong h^2/2r=h^2F/[2(n-1)].$$

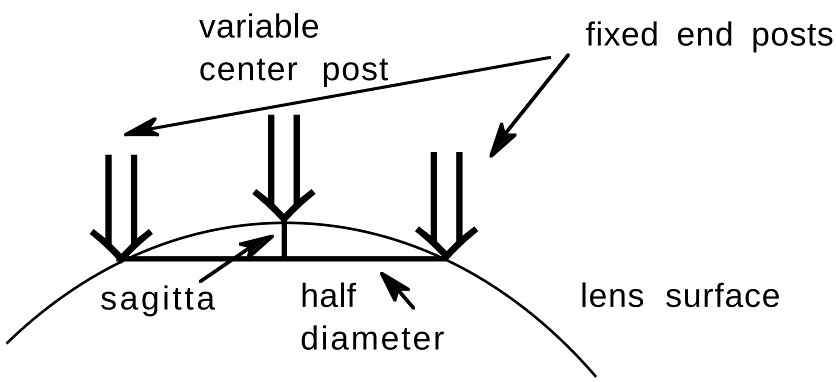
The sagitta formula is the basis of operation of a lens gauge. If a lens is made of glass or plastic of index n, then its power is

$$F=(n-1)/r.$$

If the sagitta of the surface is s with half diameter h, the surface power is

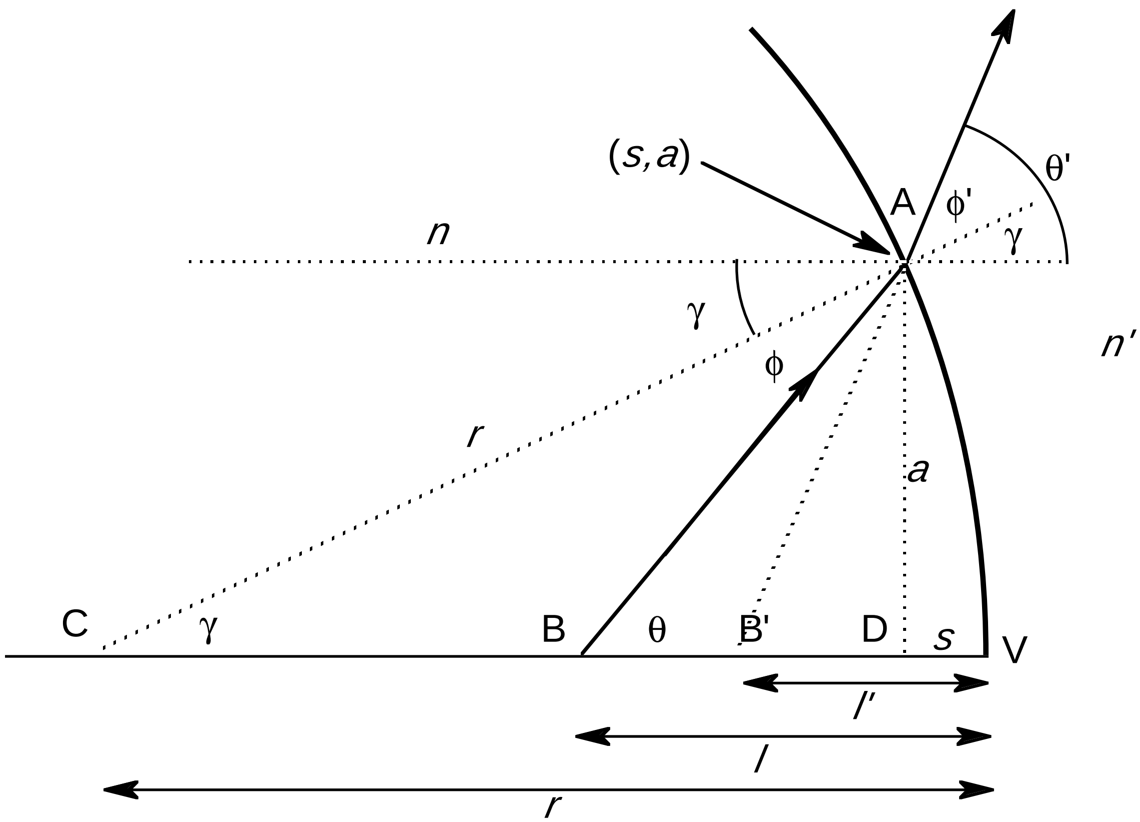
$$F=2(n-1)s/h^2.$$

The three legs of a lens gauge measure sagitta and half diameter, as shown below. The fixed outer posts are separated from the plunger by a known half diameter, usually about one centimeter. When placed on a surface the plunger moves, the amount of movement being the sagitta. This sagitta measurement is then converted mechanically to a power reading. Lens gauges are invariably calibrated for crown glass of index 1.523. Lens gauges are used to measure the parameters of spectacle lenses.



DERIVATION of the FUNDAMENTAL PARAXIAL EQUATION

But wait, we almost forgot to *derive* the fundamental paraxial equation!! We do so by deriving a set of equations that trace rays from an axial point B through a curved surface. The geometry of the meridional ray trace is shown below.



Assume we're given the radius of curvature of the surface, r, the indices of refraction on either side of the surface, n and n', and l, the distance of source B from vertex V, and we want to find out what happens to a ray that leaves B at angle θ. In particular we want to find out where it hits the curved surface, a point (s,a) measured in Cartesian coordinates centered on V, and what angle θ' it makes after refraction. We can do that by using geometry and Snell's law.

We can find ϕ, the angle which the ray makes with the surface normal by applying the law of sines to triangle ABC with the result

$$|r|/\sin(180°-|\theta|)=(|r|-|l|)/\sin|\phi|.$$

Recalling the sign convention, in the above diagram r, l, θ, $\phi<0$ so the preceding equation becomes

$$-r/\sin(180°+\theta)=(-r+l)/\sin(-\phi).$$

Since $\sin(180°+\theta)=-\sin\theta$ and $\sin(-\phi)=-\sin\phi$, this becomes, after some rearranging,

$$\phi=\arcsin\{[1-(l/r)]\sin\theta\}. \qquad (1)$$

From Snell's law, the angle of refraction, ϕ', is given by

$$\phi'=\arcsin[(n/n')\sin\phi]. \qquad (2)$$

Since the sum of the angles of a triangle is 180°, from triangle ABC,

$$\gamma+|\phi|+(180°-|\theta|)=180°$$

and simplifying, bearing in mind the sign convention,

$$\gamma=\phi-\theta. \qquad (3)$$

With respect to the optic axis, then, the refracted ray makes an angle θ' and since the whole equals the sum of its parts, $|\theta'|=|\phi|+\gamma$. Again bearing in mind sign conventions, this becomes

$$\theta'=\phi'-\gamma. \qquad (4)$$

Now we have to find the point (s,a) at which the ray intersects the refracting surface. From triangle ACD, $\cos\gamma=(|r|-|s|)/|r|$. solving for s and simplifying,

$$s=r(1-\cos\gamma). \qquad (5)$$

Again from triangle ADC, $\sin\gamma = a/|r|$, so a is given by

$$a = -r\sin\gamma. \qquad (6)$$

We can now find θ and (s,a), but what we're usually most interested in is the position of B', the point from which an observer will see light rays from B **apparently** coming. Point B' is a distance l' from the vertex. From triangle AB'D, $\tan\theta' = a/(|l'|-|s|)$ so that

$$l' = s + a/\tan\theta'. \qquad (7)$$

Summarizing these equations, then, we have

$$\phi = \arcsin\{[1-(l/r)]\sin\theta\}. \qquad (1)$$

$$\phi' = \arcsin[(n/n')\sin\phi]. \qquad (2)$$

$$\gamma = \phi - \theta. \qquad (3)$$

$$\theta' = \phi' - \gamma. \qquad (4)$$

$$s = r(1-\cos\gamma). \qquad (5)$$

$$a = -r\sin\gamma. \qquad (6)$$

$$l' = s + a/\tan\theta'. \qquad (7)$$

Equations (1)-(7) allow us to trace the ray **exactly** through the surface by solving eac equation in turn. This is called trigonometric ray tracing. It can be done by hand or with a pocket calculator, but nowadays it's easiest with a microcomputer. The equations can be programmed into BASIC or some other high level language, but it's easier to lay them out on a spreadsheet. A spreadsheet layout is shown below which calculates l' for several angles as well as calculating the paraxial approximation to l'. The first five rows, shown in boldface, are input. The numbers in the other grid positions are output generated by programming equations (1)-(7). Note the deviation from the paraxial result as $|\theta|$ becomes larger, just as we'd expect.

Trigonometric ray tracing of image position for refraction by a single spherical surface:
(All angles are in degrees and distances in meters.)

n=	**1.00**				
n'=	**1.50**				
r=	**-1.00**				
l=	**-0.50**				
θ=	**1.00**	**5.00**	**10.00**	**20.00**	**40.00**
ϕ=	0.5000	2.4976	4.9809	9.8466	18.7472
ϕ'=	0.3333	1.6648	3.3183	6.5463	12.3723
γ=	-0.5000	-2.5024	-5.0191	-10.1534	-21.2528
θ'=	0.8333	4.1672	8.3374	16.6998	33.6250
s=	0.0000	-0.0010	-0.0038	-0.0157	-0.0680
a=	-0.0087	-0.0437	-0.0875	-0.1763	-0.3625
l'=	-0.6000	-0.6002	-0.6008	-0.6033	-0.6131

Paraxial approximation:

L'=	-2.5000
l'=	-0.6000

A closed form solution for l' may be written, but in general it is tremendously cumbersome. But a very useful result (which we've already seen, of course) may be obtained in the <u>paraxial</u> limit where we assume all angles are small. In that case the equations become

$$\phi \cong [1-(l/r)]\theta. \tag{1'}$$

$$\phi' \cong (n/n')\phi. \tag{2'}$$

$$\gamma = \phi - \theta. \tag{3'}$$

$$\theta' = \phi' - \gamma. \tag{4'}$$

$$s \cong 0. \tag{5'}$$

$$a \cong -r\gamma. \tag{6'}$$

$$l' \cong s + a/\theta'. \tag{7'}$$

Let's solve (1')-(7') successively to obtain l' given l, r, n, n', and θ. Combining (2') and (1'),

$$\phi' \cong (n/n')\phi = (n/n')[(r-l)/r]\theta. \tag{2''}$$

Combining (3') and (1'),

$$\gamma = \phi - \theta = [(r-l)/r]\theta - \theta = -l\theta/r. \tag{3''}$$

Combining (4'), (2''), and (3''),

$$\theta' = \phi' - \gamma = (n/n')[(r-l)/r]\theta - (-l\theta/r) = [(n/n')(r-l)+l](\theta/r). \tag{4''}$$

Combining (6') and (3''),

$$a \cong -r\gamma = -r(-l\theta/r) = l\theta. \tag{6''}$$

Finally, combining (7'), (5'), (6''), and (4''),

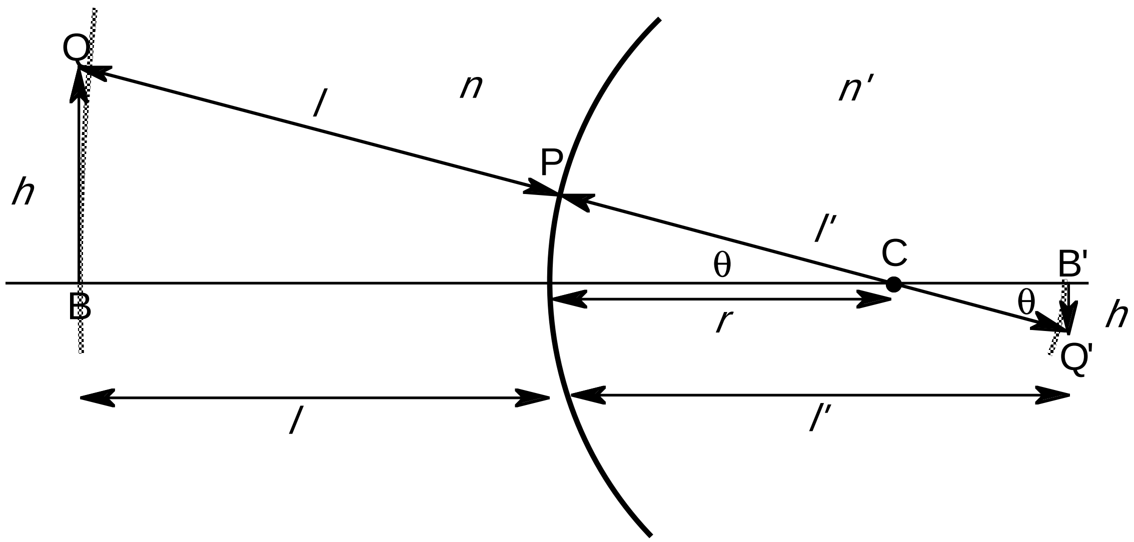
$$l' = a/\theta' = \{l\theta\}/\{[(n/n')(r-l)+l](\theta/r)\} = lr/\{(n/n')r + [(n'-n)/n']l\}. \tag{7''}$$

Equation (7") becomes more tractable if we take its reciprocal and multiply by n' to obtain

$$n'/l' = n/l + (n'-n)/r.$$

And that is the, by now, familiar fundamental paraxial equation for a spherical surface.

Use the diagram below to derive the magnification expression.



In the diagram, the surface of radius r separating media of indices n and n' forms the image of height h' of an object of height h. Points B and Q at the bottom and top of the object are conjugate to points B' and Q', respectively, on the image. Because the object and image are small, i.e. angle θ is small, they lie nearly along circular arcs centered at C, the center of curvature of the surface. (These arcs are indicated in gray in the diagram.) Thus corresponding points on the object and image are conjugate satisfying the fundamental paraxial equation in the form

$$n'/l' = n/l + (n'-n)/r. \tag{8}$$

From the geometry of the diagram it is apparent that

$$h'/h = (l'-r)/(l-r). \tag{9}$$

(Remember those sign conventions!) Multiply the top and bottom of the right hand side of (9) by $nn'/ll'r$ to get, after simplification,

$$m=[(n/l)/(n'/l')]\{[(n'/r)-(n'/l')]/[(n/r)-(n/l)]\}.$$

But from the fundamental paraxial equation, (8), the expression in curly brackets is just 1, so the last equation reduces to

$$m=(n/l)/(n'/l')=L/L'.$$

This is the familiar result of paraxial theory.

There now!! that wasn't so bad, was it?

