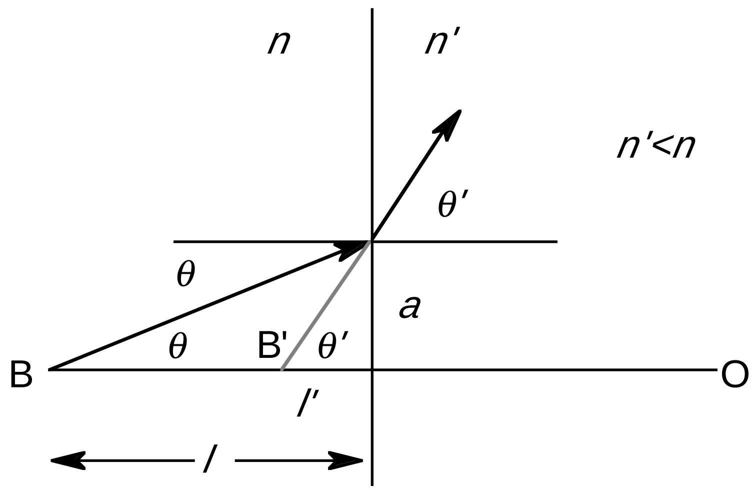
REFRACTION by PLANE SURFACES

Apparent Depth

Suppose we have an object B in a medium of index n which is viewed from a medium of index n'. If $n'<n$, the diagram of this would be like that below.



What does the observer O, who is looking along the normal to the interface, see when he looks at object B? The diagram shows a typical ray leaving B. It makes an angle θ with the normal when it his the interface. It leaves the interface at angle θ' where from Snell's law,

$$\sin\theta' = (n/n')\sin\theta. \tag{1}$$

If the object is a distance l from the interface and the ray intersects the interface at a point a distance a from the line of sight, by trigonometry

$$\tan\theta = a/l. \tag{2}$$

If the extension of the refracted ray intersects the optical axis at l', then from trigonometry

$$\tan\theta' = a/l'. \tag{3}$$

Now let's assume that the rays that actually enter the observer's eye are very nearly parallel to the optic axis, the so-called paraxial approximation. Then from the small angle approximation $\theta \cong \sin\theta \cong \tan\theta$ and $\theta' \cong \sin\theta' \cong \tan\theta'$ so equations (1), (2), and (3) become

$$\theta' = (n/n')\theta. \tag{1'}$$

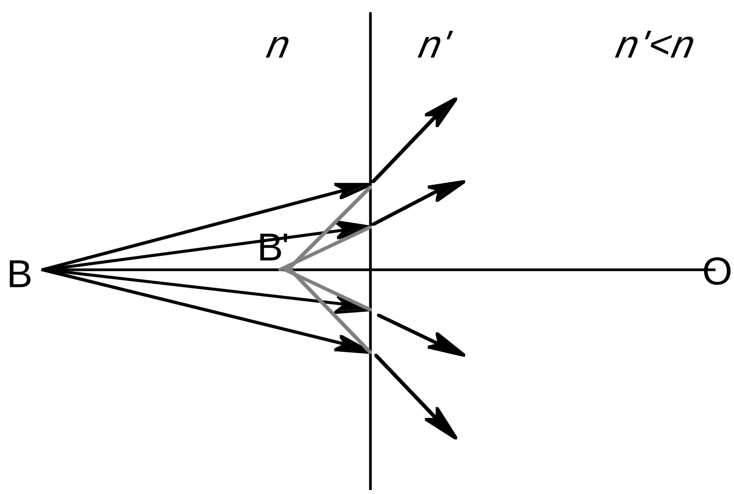
$$\theta = a/l. \tag{2'}$$

$$\theta' = a/l'. \tag{3'}$$

Substituting (2') and (3') into (1') gives

$$n'/l' = n/l. \tag{4}$$

Note that the result is independent of *a*, so all rays from B intersect the axis at the same point B'. That would look like the following.



To the observer all the light seems to be coming from point B' a distance *l'* beneath the interface, not from point B a distance *l* below the interface. B' is the image of B and equation (4) is the fundamental paraxial equation for a plane surface relating object and image distances. Note, this only happens for paraxial rays. If rays make large enough angles with the interface, the refracted rays do not all intersect at point B', and the image is somewhat blurred or aberrated.

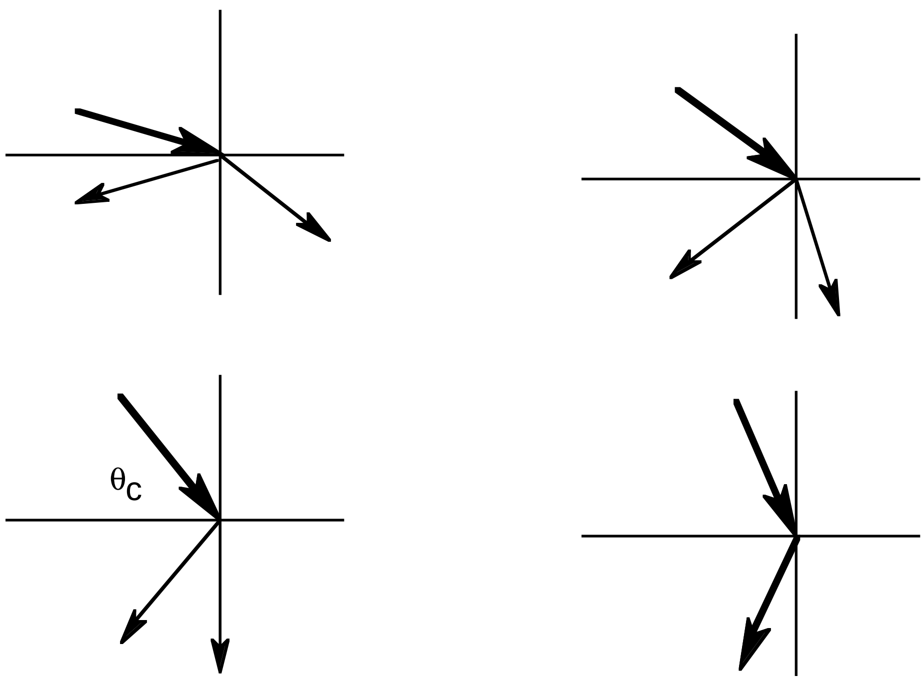
Example: A penny is at the bottom of a swimming pool three meters deep. How far below the surface of the water does the penny seem to be to a diver looking down on it.

Solution: In this case $l=3\text{m}$, $n=1.33$ (the index of water), and $n'=1$ (the index of air) so the apparent depth of the penny is

$$l'=(n'/n)l=(1/1.33)(3\text{m})=2.26\text{m}.$$

Note that the coin looks closer to the surface than it really is.

Total Internal Reflection



In going from denser to rarer medium, part of the energy of a ray is reflected, part refracted <u>away</u> from the normal to the surface. At one particular incident angle θ_c, the refracted ray makes a 90° angle with the normal. For any angle of incidence greater than θ_c, all the energy of the ray is reflected. The angle θ_c, is called the critical angle. It satisfies Snell's law $\sin\theta_c=n'\sin 90°=n' \Rightarrow \sin\theta_c=n'/n$.

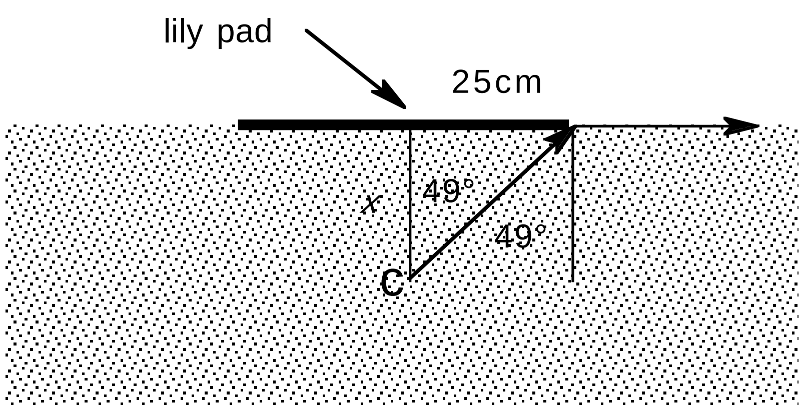
Critical angle has a number of practical applications in producing a mirror surface without silvering. It is used in fiber optic probes, diamonds, a variety of reflecting prisms, etc.

Example: What is the critical angle for light going from water to air?

Solution: θ_c=arc sin(n'/n)=arc sin(1/1.33)=49°.

Example: A heron is hunting a frog hiding beneath a lily pad of 50cm diameter. What is the maximum distance the frog may go beneath the lily pad before the heron can see him?

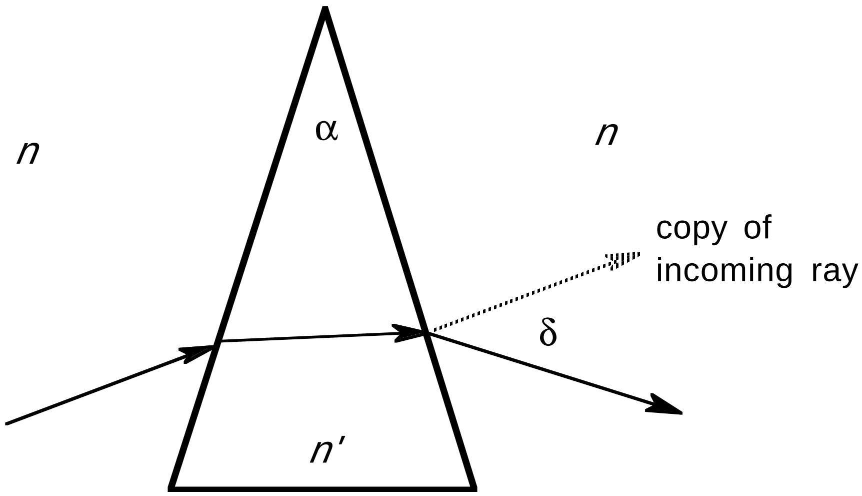
Solution:



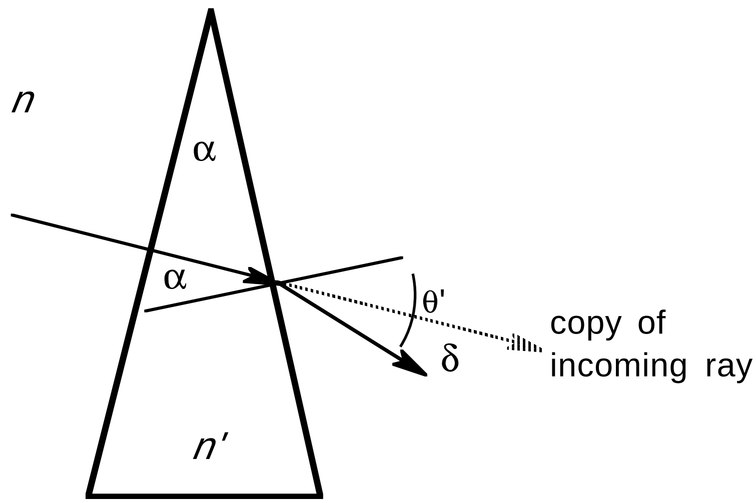
The frog will remain invisible to the heron so long as rays from the frog's body are totally reflected at the air-water interface. The maximum depth for which this will be true is beneath the *center* of the lily pad. At that maximum depth, rays from the frog will strike the edge of the lily pad at the critical angle which we calculated to be 49° in the previous example. From the diagram that distance, x, is given by 25cm/x=tan49° so x=25cm/tan49°=21.9cm.

Prisms

Let a ray of light strike a wedge of refractive material, a prism. The wedge has apical angle α and index n' and is surrounded by a medium of index n. If, as is always the case in realistic problems, the prism is more optically dense than the surrounding mirror, the ray is bent toward the prism base as it passes through each prism face. The question of interest is, what is the angle δ through which the incoming ray is turned?



Calculating δ is, in general, a complicated geometric problem. But in ophthalmic optics, only thin prisms, those of small apical angle, are important and only rays incident nearly normally are considered. That simplifies the problem a lot. The diagram below shows such a prism.



Let's simplify the calculation by having the ray incident normally on the first face of the prism. It is not, then deviated by the first face and strikes the second face such that it makes an angle α with the normal. It leaves the prism at an angle θ' with the normal. From Snell's law,

$$n'\sin\alpha = n\sin\theta' \text{ or } n'\alpha \cong n\theta'. \tag{5}$$

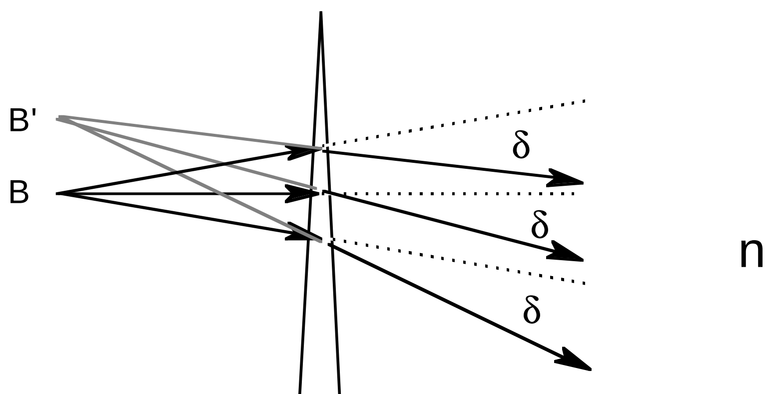
But the angle of deviation δ is related to θ' by

$$\theta' = \alpha + \delta. \tag{6}$$

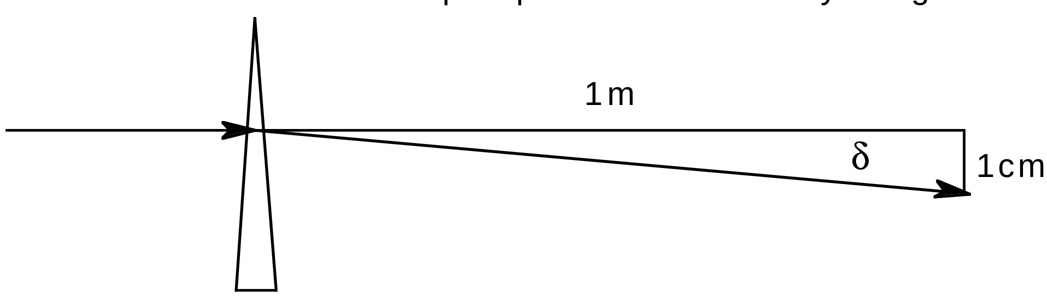
Eliminating θ' from (5) and (6),

$$\delta=[(n'/n)-1]\alpha. \tag{7}$$

Even though (7) was derived for a nearly normally incident ray, it is true for any ray incident nearly normally. That means that all such rays are deviated through exactly the same angle. If the rays come from an object B, as in the diagram below, an observer sees them as if they came from an image B', displaced vertically from B but not displaced longitudinally.



Ophthalmic prisms are thin prisms. They are measured in prism diopters of prism power. A prism which causes a deviation of a ray of light of 1cm per meter travelled by the ray has one prism diopter of prism power. The diagram below shows how a one diopter prism deviates a ray of light.



From the diagram, the angle of deviation is δ(radians)=1cm/1m=0.01. If the prism has Δ prism diopters of power, it moves the beam Δ centimeters for every meter travelled and $\delta=\Delta$cm/1m=0.01Δ or

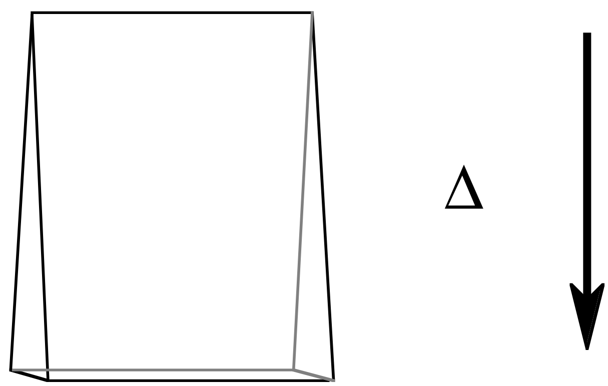
$$\Delta=100\delta(\text{radians})=100(n'-1)\alpha(\text{radians}) \tag{8}$$

since n=1 for prisms in air.

Example: What is the power of a crown glass prism of apical angle 4°?

Solution: $\alpha=4°=(\pi/180°)x4°=0.070$ radians and $n'\doteq 1.523$. Plugging into (8) gives $\Delta=100(1.523-1)(0.070)=3.65$prism diopters$\cong 4$ p.d.

Superposition of Thin Prisms

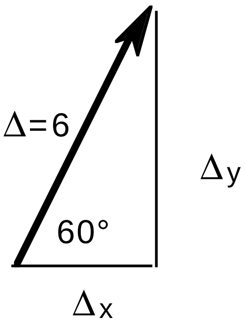


As shown in the diagram above, an ophthalmic prism may be represented as a vector with the magnitude of the prism and direction from apex-to-base. It would have been nicer if the opposite direction since the image moves toward the apex and the prism sort of points that way, but that just isn't how it's done.

Superimposed prisms may be combined by just adding the corresponding vectors and they may be resolved into components, just like any other vectors.

Example: The apex-to-base line of a 6p.d. prism makes an angle of 60° with the x-axis. What is the prism power in the horizontal and vertical meridians?

Solution: Draw the vector representation of the prism. From trigonometry Δ_x=6cos60°=3p.d., Δ_y=6sin60°=5.2p.d.



The whole vector could be written in component form as **Δ**=(3.0, 5.2).

Example: What prism is equivalent to a 5p.d. base @ 45° prism superimposed on a 3p.d. base down prism?

Solution: Resolve each prism into components. The vector corresponding to the first prism is $\mathbf{\Delta}_1$=(5cos45°,5sin45°)=(3.5,3.5). The vector corresponding to the second prism is $\mathbf{\Delta}_2$=(0,-3). The total prism is the vector sum,

$$\mathbf{\Delta}=\mathbf{\Delta}_1+\mathbf{\Delta}_2 =(3.5+0,3.5-3)=(3.5,0.5).$$

The magnitude of the resultant prism is

$$\Delta=\sqrt{[(3.5)^2+(0.5)^2]}=3.54\text{p.d.}$$

and the angle is θ=arc tan(0.5/3.5)=8°8'.