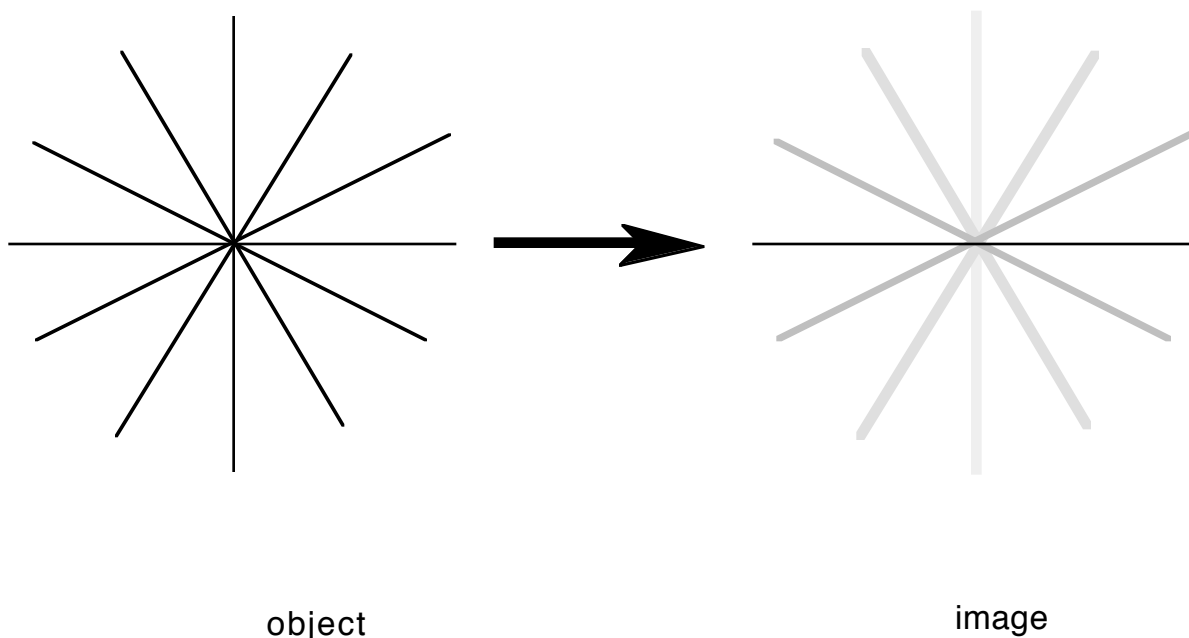# TORIC LENSES

Astigmatism

In astigmatism, different meridians of the eye have different refractive errors.   This results in horizontal and vertical lines being focused different distances from the retina. Here, for example, is the image of a fan dial chart formed by an astigmatic eye.

object   image

The amount of astigmatism is characterized by the dioptric difference of the two meridians that differ the most. These meridians are invariably at right angles to one another. Usually they are roughly horizontal and vertical, but sometimes they may be oblique. While the myope can at least see near objects clearly and the hyperope may, through the exercise of accommodation, see distant objects clearly, the astigmat sees objects at all distances as being blurred.

The most common cause of astigmatism is a cornea with a surface shaped more like that of a football than a basketball. High astigmatism is almost
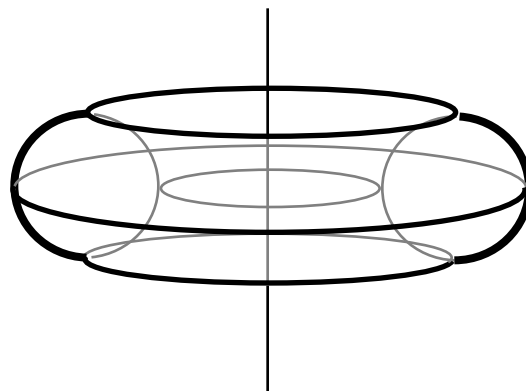
always corneal. Smaller contributions come from <u>internal astigmatism</u> thought to be due mainly to the crystalline lens.

Astigmatism can be and usually is combined with myopia or hyperopia.

Astigmatism can be corrected with <u>cylinder</u> lenses. In practice it is corrected with lenses which have <u>toric</u> surfaces.
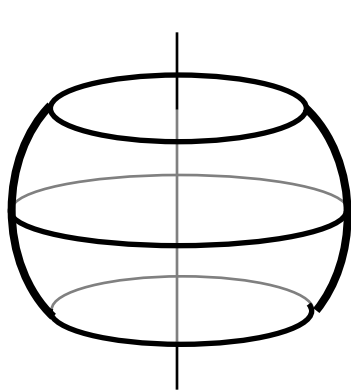
barrel form   tire form

A toric surface is generated mathematically by rotating a circular arc around an axis that does **not** go through the center of the circle. (If we rotated it around the center of the circle we'd just get a sphere.)   This generates surfaces in barrel form or tire form (donut form) as shown above.

The prescription for a toric lens is written as if it were a superposition of a spherical lens (which has the same prescription in all meridians) and a cylinder lens (which has power in only one meridian). For example

$$-5.00/-2.00x175$$

corresponds to five diopters of myopia on top of two diopters of astigmatism with the correcting cylinder lens oriented at 175°.

Power Cross

Spectacle lenses consist of one spherical surface and one toric surface. Prescriptions for spectacle lenses are, however, written as a superposition of spherical and cylindrical lenses. There are three ways to write a spectacle prescription; plus cylinder form (favored by M.D.'s), minus cylinder form (favored by optometrists), and cross cylinder form (used, sometimes, in fabrication labs). It's easiest to understand these forms and their relationship by reference to a power cross like the one below.
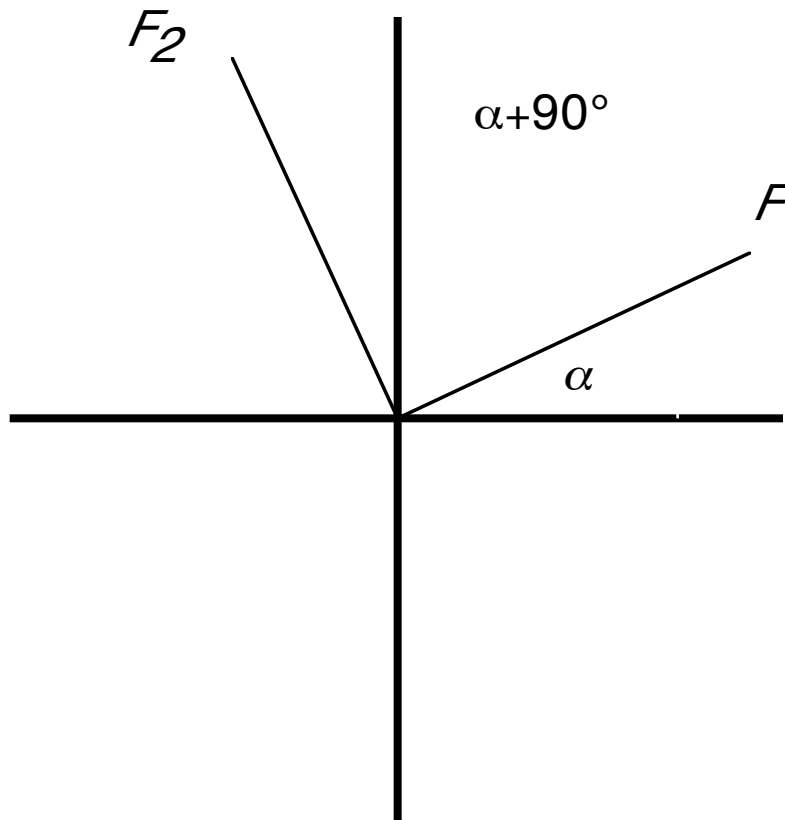
This cross represents a lens surface as viewed from the doctor's point of view, i.e. looking straight at the patient. The lens has two principal meridians (meridians of circular cross-section), with powers F_1 and F_2. Their orientations are shown on the diagram.

We could specify the prescription for the lens in the diagram by writing
$$F_1 @ \alpha / F_2 @ (\alpha \pm 90°).$$

(1)

© W. F. Long, 1992

That is how keratometric findings are typically written, but lens powers are never written in this form. Recall that for a pure cylinder lens of power F_C and axis α the prescription can be written as $F_C\,\text{x}\,\alpha$ or $F_C@(\alpha\pm90°)$. We can think of the lens in the diagram as a superposition of two cylinder lenses and write (1) as

$$F_S\text{x}\alpha\,/(F_S+F_C)\text{x}(\alpha\pm90°).$$

$$(2)$$

Equation (2) is the <u>cross cylinder form</u>. In diagram terms it sets up like below:

$$F_1\text{x}(90°\pm\alpha)\qquad\qquad\qquad F_2\,\text{x}\alpha\,.$$
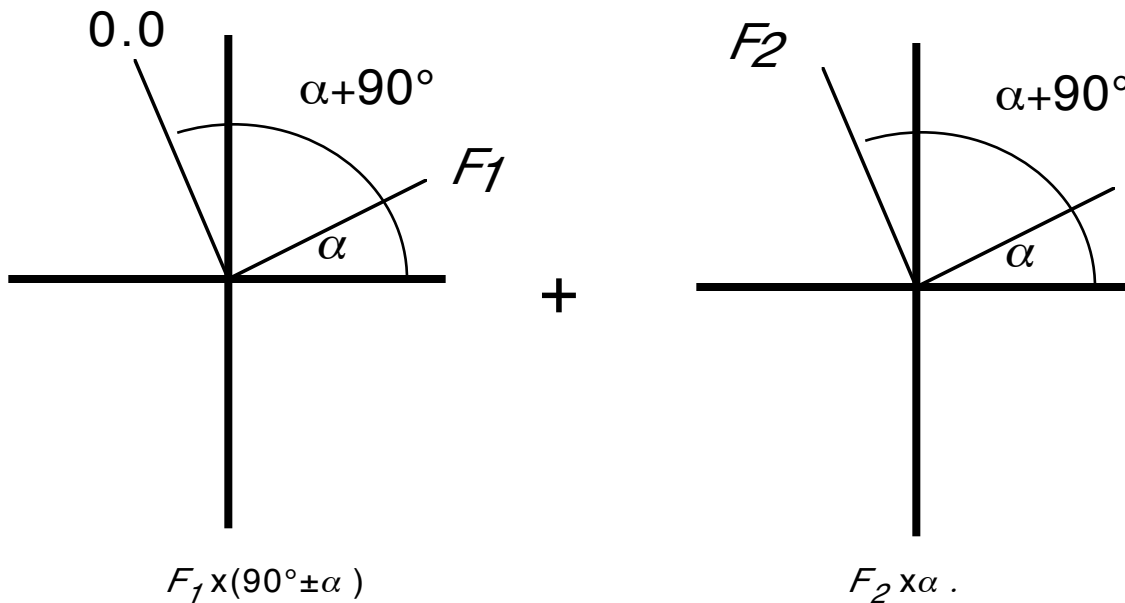
Since you can add sphero-cylinder lenses meridian by meridian, as long as the principal meridians are the same, you can see that this form gives the correct prescription.

Another, more common way of writing the prescription is in sphero-cylinder form in which the prescription is represented as a spherical lens of power F_S superimposed on a cylinder lens with power and axis $F_C\text{x}\alpha$. This is written in the form

$$F_S\,/F_C\text{x}\alpha\,.$$

$$(3)$$

© W. F. Long, 1992

In diagrams, equation (3) is represented as

FS
$\alpha+90°$
FS
α
F_S

+

FC
$\alpha+90°$
α
$F_C\text{x}\alpha$
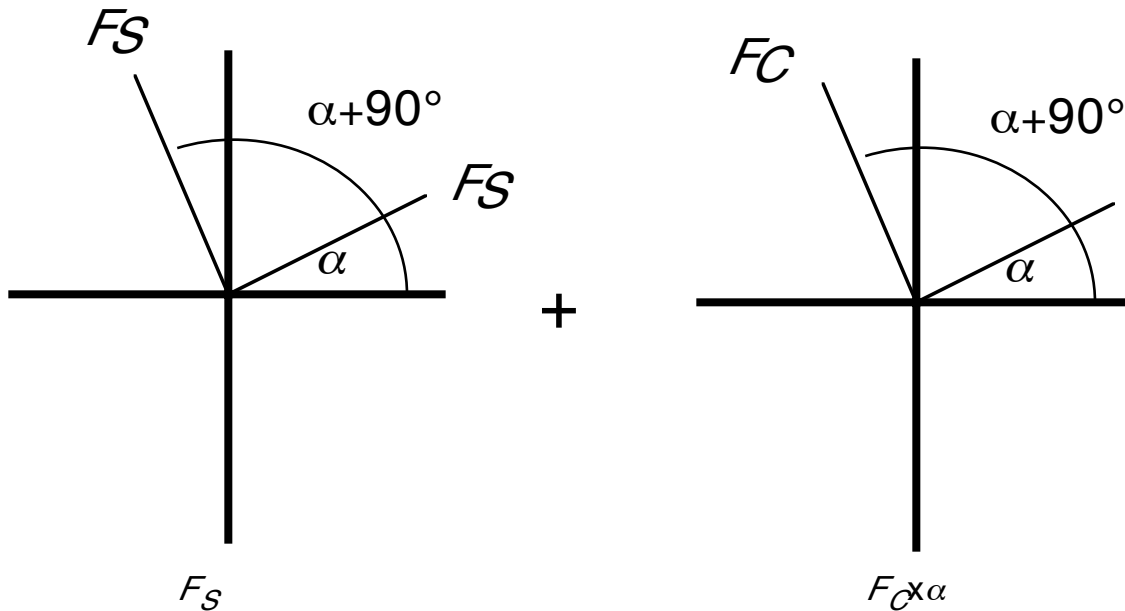
The cylinder power can be positive or negative. The sign of the cylinder determines whether the prescription is in plus or minus cylinder form.

Now the problem is how to relate the sphero-cylinder form (3) to the crossed cylinder form (2).   The key to doing this is in the diagrams. Clearly, by equating powers in corresponding meridians, $F_1 = F_S + F_C$ and $F_2 = F_S$.   The cross cylinder form corresponding to the sphero-cylinder form is thus,

$$(F_S + F_C)\text{x}(\alpha \pm 90°)/F_S\text{x}\alpha .$$

(4)

A given spectacle prescription can be written in any of these forms. The problem is converting among them. Here are three equivalent forms of a prescription.

$$F_S / F_C\text{x}\alpha$$
$$(F_S + F_C)/(-F_C)\text{x}(\alpha \pm 90°)$$
$$F_S\text{x}\alpha/(F_S + F_C)\text{x}(\alpha \pm 90°).$$

(5)

If $F_C > 0$, these are, respectively, the plus cylinder, minus cylinder, and

cross cylinder forms. Note that $\alpha \pm 90°$ is taken such that $180° \geq \alpha \pm 90° > 0$. In cross cylinder form it is customary to write the meridian for the smaller angle first.

Let's try it with a numerical example. Here are three equivalent forms of the same prescription:

$$+3.00-1.00 \times 030$$
$$+2.00+1.00 \times 120$$
$$+2.00 \times 030 / +3.00 \times 120$$

Base Curves

Base curve is a term which is used in a number of confusing ways. For an optometrist right now it may be considered to be a single power which specifies lens form, usually the front surface power of a lens. To add to the confusion, in communicating with labs about base curves, one generally refers to the lens clock reading on the front surface curve with a lens gauge calibrated for crown glass, regardless of the actual index of the material of the lens.

A notation that takes into account lens form is as follows:

(front surface formula)/(back surface formula).

Customarily the surface formulas are given in cross-cylinder form, the power of the lowest magnitude coming first, for example,

$$+4.00/-2.00x110 \Rightarrow (+6.00x110/+8.00x020)/(-4.00).$$

If the front surface is toric, the lens is said to be ground in plus cylinder form. If the back surface is toric, the lens is said to be ground in minus cylinder form.

Usually base curves are chosen by the lens designer to minimize certain aberrations. The optometrist might want to specify base curves in three cases:

☞ To get a back surface curve deep enough to provide eyelash clearance.

☞ To get a particular magnification of the retinal image.

☞ To duplicate the performance of a lens to which the patient has become accustomed.

© W. F. Long, 1992

The Astigmatic Pencil

Toric or spherocylinder lenses produce a characteristic   astigmatic pencil of light when placed behind a point source.
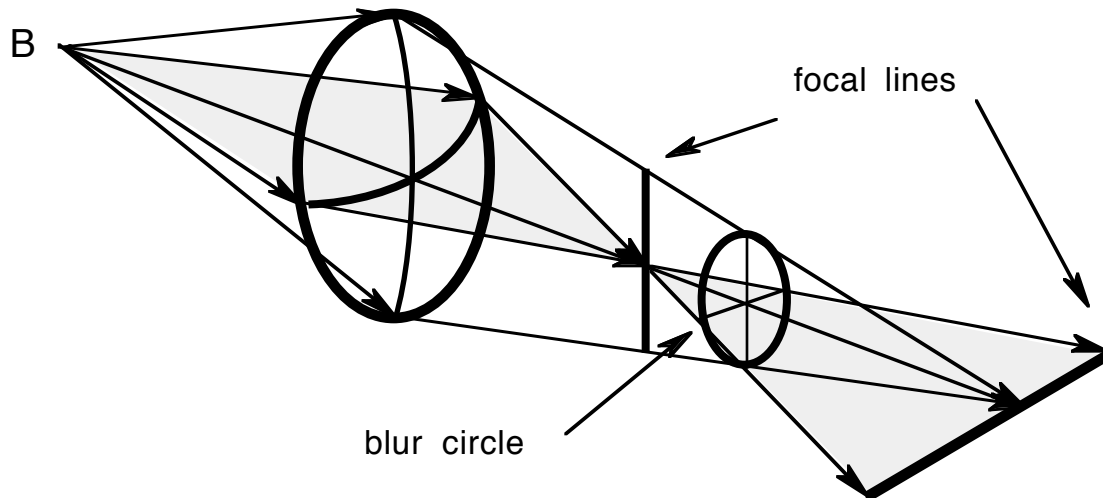
The lens produce vertical and horizontal line foci separated by the Sturm's interval. The positions of the line foci may be calculated by applying the fundamental paraxial equation in each of the two principal meridians remembering that the vertical line focus is formed when rays in the horizontal plane come to a focus and the horizontal line focus is formed when rays in the vertical plane come to a focus. Dioptrically midway between the two line foci the astigmatic pencil assumes a circular cross-section, the so-called circle of least confusion.

Example: Find the positions of the line foci and blur circle and the interval of Sturm for a toric lens with +5.00D in the horizontal meridian and +8.00D in the vertical meridian when imaging a point object one meter in front of the lens.

Solution: The incoming vergence is L=1/(-1m)=-1.00D. The outgoing vergence in the horizontal meridian is +5.00D-1.00D=+4.00D. This corresponds to a distance of 1/(+4)=+25cm, where the vertical line focus is formed. In the vertical meridian, outgoing vergence is +8.00D-1.00D=+7.00D. This corresponds to a distance of 1/(+7.00)=+14.3cm, where the horizontal line focus is formed. The interval of Sturm is just 25cm-14.3cm=10.7cm. The vergence going to the blur circle is (+4+7)/2=+5.5D so the blur circle is 1/(+5.5D)=+18.2cm behind the lens.

© W. F. Long, 1992

Equivalent Sphere

The spherical lens with a power equal to that of the average of the two principal meridians is the <u>equivalent sphere</u>. For a prescription S/Cxθ, the equivalent sphere is just $S + C/2$.

From the discussion of the astigmatic pencil, it is clear that a spherical lens with the power of the equivalent sphere would focus light from a remove object in the plane of the circle of least confusion. Placed before the eye, and equivalent sphere lens would place the circle of least confusion at the retina.

Power in Oblique Meridians of a Spherocylinder Lens

The meridians of maximum and minimum power of a spherocylinder lens, the <u>principal meridians</u>, have spherical cross-sections. But what about oblique meridians? Such meridians do **not** have spherical cross-section and, hence, have no real optical power. But they **do** have an oblique power that is useful in certain contexts. If lens formula is S/Cxθ, then the power in a meridian at angle ϕ is

$$F_\phi = S + C\sin^2(\theta - \phi).$$

© W. F. Long, 1992

Prismatic Effects in Spherocylinder Lenses

When a ray passes through an optical system its trajectory is bent, just as it would be with a prism. The amount of deviation of the ray increases the farther the ray is from the optic axis. We can look at these off-axial points as if they were prisms, the power of which varies with their position. As we have seen, the amount of prism at a point a distance c from the optical center of a spherical lens of power F is given by

$$\Delta = -c/f' = -Fc.$$

where F and c are given in diopters and meters, respectively, and Δ is in prism diopters. This is known as <u>Prentice's Rule</u>. The minus sign indicates that the prism has its base **toward** the optic axis for a lens of positive power and **away** from the optic axis for a lens of negative power. The distance from the axis to the off-axial point and prism power can be treated as a vector quantities $\boldsymbol{c}$ and $\boldsymbol{\Delta}$, in which case Prentice's Rule becomes

$$\boldsymbol{\Delta} = -F\boldsymbol{c}.$$

Here $\boldsymbol{\Delta}$ is a vector running from the apex to the base of the prism.

© W. F. Long, 1992

For a cylinder lens, Prentice's rule only applies to the meridian in which there is power. The diagram below shows the power cross for a cylinder with axis 90°. We wish to know the amount of prism at a point a distance **c** from the optic axis of the cylinder lens.

$$0.00$$
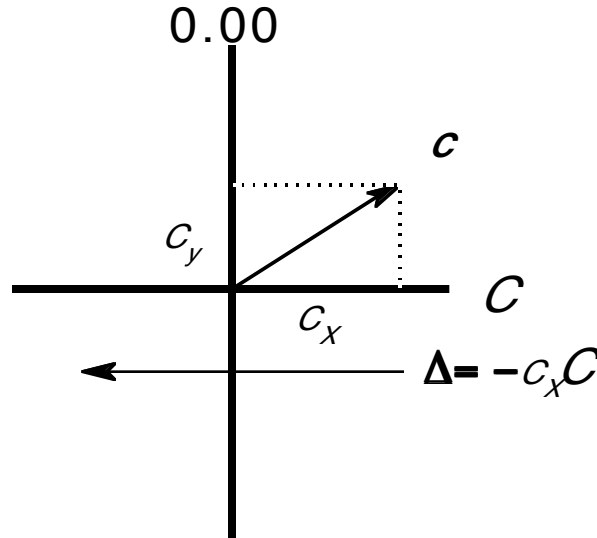
The power of the prism depends only on the *x* component of the vector **c** and its direction lies along the power meridian of the prism.

Since any spherocylinder lens can be considered a combination of two cylinders at right angles, the principal above may be used to determine the prism at an arbitrary point on a spherocylinder lens. For a lens with formula S/Cxα, the prism Δ at a point **c** is given by the following vector equation:

$$\Delta=(\Delta_x,\Delta_y)=-(Hc_x+Qc_y, Qc_x+Vc_y),$$

where

$$H \equiv S + C\sin^2\alpha,$$
$$V \equiv S + C\cos^2\alpha,$$
$$Q \equiv -C\sin\alpha\cos\alpha.$$

Here H is the power of the lens in the horizontal meridian and V is the power of the lens in the vertical meridian. The quantity Q has no ready physical interpretation but is related to the obliquity of the cylinder.

© W. F. Long, 1992

Example: What prism is encountered at a point 5mm in and 15 mm up on a right lens of prescription -6.00-2.00x140?

Solution: In this problem the vector c=(+0.5, +1.5) and H, V, and Q are

$$H=-6-2\sin^2 140°=-6.826D$$
$$V=-6-2\cos^2 140°=-7.174D$$
$$Q=-(-2\sin 140°\cos 140°)=-0.985D.$$

hence

$$\Delta_x=-Hc_x-Qc_y=6.826(0.5)+0.985(1.5)=+4.9 \text{p.d.},$$
$$\Delta_y=-Qc_x-Vc_y=0.985(0.5)+7.174(1.5)=+11.3 \text{p.d.}$$

The final answer, then, is 5p.d. BI and 11p.d.BU.

Example: What is the prism at a point 5mm out and 7mm down for the right lens of a patient with prescription +5.00-2.00x180.

Solution: When the cylinder axis 180° or 90°, Q=0 and $\Delta=(\Delta_x, \Delta_y)=-(Hc_x, Vc_y)$. In other words, Prentice's rule is just applied along the horizontal and vertical meridians. In this problem the power cross is
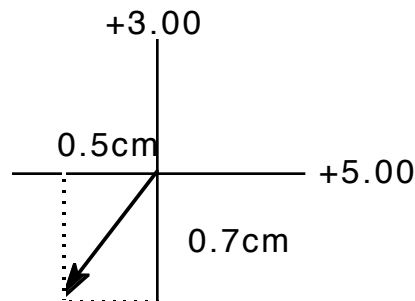
so that $\Delta_x=-(-0.5\text{cm})(+5.00)=+2.5$p.d., BR or BI and $\Delta_y=-(-0.7\text{cm})(+3.00)=+2.1$p.d., BU.

© W. F. Long, 1992

In laying out a lens for edging, the optical center of the lens is moved so that the major reference point of the lens will sit in front of the patient's pupil. In <u>decentering</u> a lens a distance *d*, the prism achieved is that which is at a point *c=-d* from the optical center of the lens. The prism achieved by decentering a lens is then given by

$$\Delta=(\Delta_x, \Delta_y)=(Hd_x+Qd_y,\ Qd_x+Vd_y).$$

A frequent problem is determining the decentration of a lens to obtain a given amount of prism. That can be obtained by inverting the equation above with the result

$$d=(d_x,\ d_y)=(1/D)(V\Delta_x-Q\Delta_y,\ -Q\Delta_x+H\Delta_y),$$

where

$$D\equiv S(S+C).$$

Example: The Rx of a patient is -5.00/-2.00x025, O.S. Find the decentration to give a prism of 3p.d. BD and 4p.d. BO.

Solution: For this lens

$$H=-5-2\sin^2 25°=-5.357D$$
$$V=-5-2\cos^2 25°=-6.643D$$
$$Q=-(-2\sin 25°\cos 25°)=+0.766D$$
$$D=(-5)(-7)=+35$$

The required prism is D=(+4, -3). Substituting in we get

$$d_x=[(-6.643)(+4)-(+0.766)(-3)]/35=-0.69cm=-6.9mm$$
$$d_y=[-(+0.766)(+4)+(-5.357)(-3)]/35=+0.37cm=+3.7mm$$

or 7mm in (left) and 4mm up.

Obliquely Crossed Cylinders

Suppose two spherocylinder lenses are alligned with their optical centers coincident and then the power of the combination is read with a lensometer. Surprisingly, the combination acts like a new spherocylinder lens with cylinder and sphere power somewhere between those of the component spherocylinders. Various algorithms have been given for finding the sphere, cylinder, and axis of the combination. Here is one version derived by Thompson.

Suppose the two spherocylinder lenses have prescriptions $S_1/C_1 \times \alpha_1$, and $S_2/C_2 \times \alpha_2$ where $\alpha_2 > \alpha_1$. If both lens formulas are in the same form, the spherocylinder equivalent to the combination has the prescription $S/C \times \alpha$ where

$$C = \pm\sqrt{[C_1^2 + 2C_1C_2\cos 2(\alpha_2 - \alpha_1) + C_2^2]}$$

$$S = S_1 + S_2 + (C_1 + C_2 - C)/2$$

$$\alpha = \frac{\alpha_1 + \tan^{-1}\{C_2\sin 2(\alpha_2 - \alpha_1)/[C_1 + C_2\cos 2(\alpha_2 - \alpha_1)]\}}{2}$$

In the first of these equations the ± sign indicates that either the positive or the negative square root may be used, corresponding to plus or minus cylinder form.

Example: What lens is equivalent to the combination
(-1.00/-2.00x030)/(+4.00-3.00x050)?

Solution: Using the equations above in minus cylinder form,

$$C = -\sqrt{[(-2)^2 + 2(-2)(-3)\cos 2(50° - 30°) + (-3)^2]} = -4.71D$$

$$S = (-1) + (+4) + [(-2) + (-3) - (-4.71)]/2 = +2.86D$$

$$\alpha = 30° + \tan^{-1}[(-3)\sin 2(50° - 30°)/[(-2) + (-3)\cos 2(50° - 30°)] \div 2 = 42.08°$$

So to the nearest 1/8 diopter, the combination is equivalent to +2.875/-4.75x042.

Some applications of obliquely crossed cylinders are

☞   Over-refraction

☞   The Humphrey refractor which gives S/C_1x`180/C_2x090

☞   Javal's rule

☞   Evaluating differences in pre- and postsurgical refractions

☞   Evaluating accuracy of a refraction

☞   Evaluating degree of error in a fabrication

Sagitta Formula and Spherocylinder Lenses

Since the cross-sections in the two principal merdians of a spherocylinder lens are circular, we may apply the sagitta formula to these meridians. Recall, the sagitta formula is

$$s = r - \sqrt{(r^2 - h^2)} \cong h^2/2r.$$

This can be used to find the thickness of lenses in the principal meridians. If the sine squared formula is used for curvature in oblique meridians, the sagitta formula can be used to give approximate thicknesses in oblique meridians.

Example: A spectacle lens made of plastic of index 1.49 has Rx +3.00+5.00x090. If the lens blank is 60mm in diameter and has 2.0mm thickness at its thinnest edge, what is its thickness at its thickest edge?

Solution: The radii of curvature of the two principal meridians are
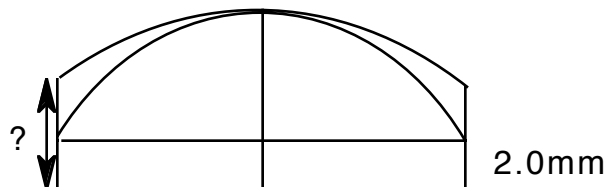
r_1=(1.49-1)/(+3.00)=0.16333m=163.33mm
r_2=(1.49-1)/(+8.00)=0.06125m=61.25mm

Since the half diameter for both meridians is 30mm, the sagittas in the two meridians are

$$s_1 = 163.33 - \sqrt{[(163.33mm)^2 - (30mm)^2]} = 2.78mm$$
$$s_2 = 61.25 - \sqrt{[(61.25mm)^2 - (30mm)^2]} = 7.85mm$$

Remember that the sagitta is the difference between center and edge thickness.

?   2.0mm

The diagram superimposes the two principal meridians. Clearly the thinnest edge thickness corresponds to the largest sagitta, 7.85mm. Thus the center thickness of the lens is 7.85mm+2.00mm=9.85mm. The thickness of the thickest edge is then 7.85mm-2.78mm=5.07mm.

Why Do Toric Lenses Work?

It is somewhat surprising that human astigmatism resolves itself into refractions which can be completely corrected by the relatively simple geometry of toric lens surfaces. Why is this so?

It is so because **all** surfaces, including the cornea and other ocular surfaces are locally toric. And toric surfaces, even when separated and with oblique axes, produce combinations which may be neutralized by toric lenses.