IMAGE PHOTOMETRY in GENERAL OPTICAL SYSTEMS

THE FUNDAMENTAL PARAXIAL EQUATION

From basic paraxial optical theory, if an object is a distance l from the first principal plane, P, of an optical system of secondary focal length f', then its image is a distance l' from the second principal plane, P', where l and l' are related by the fundamental paraxial equation

$$n/l + n'/f' = n'/l'. \tag{1}$$

The magnification of the image is given by

$$m = (n/n')(l'/l). \tag{2}$$

These equations are written using the Cartesian sign convention, i.e. distances to the right and above a reference plane are positive, those to the left and below are negative. Here n and n' are the indices of object and image space, respectively. This is illustrated in figure 1.

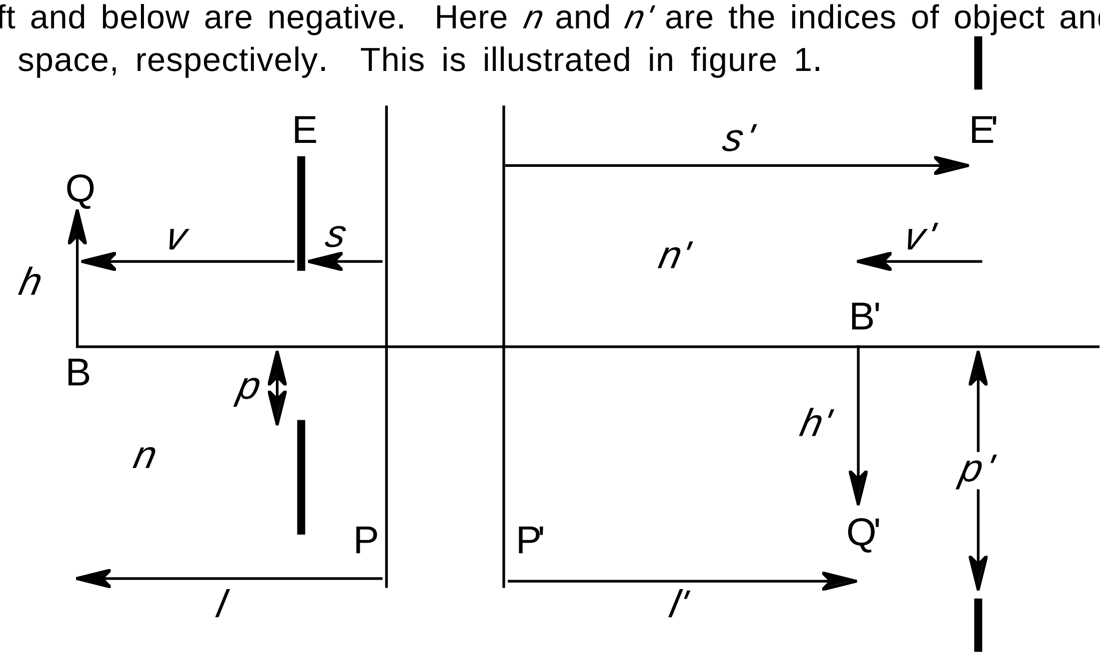

Figure 1. Coordinate systems for writing the fundamental paraxial equation in a thick lens system.

Now suppose that instead of l, we know v, the distance from the entrance pupil E to the object. How could we find the position and magnification of the image? What we'll do here is to derive a new form of the fundamental paraxial equations (1) and (2), the new forms using the entrance and exit pupils as reference planes instead of the principal planes.

The entrance and exit pupils are conjugate to one another so if s is the distance from P to E and s' the distance from P' to E', (1) and (2) give

$$n/s+n'/f'=n'/s',$$

$$p'/p=(n/n')(s'/s).$$

(Note that we're treating p and p' as signed quantities so that in figure 1 $p>0$ and $p'<0$.) From figure 1, s=PE=PB+BE=PB-EB=$l-v$, and likewise $s'=l'-v'$, so the preceding two equations become

$$n/(l-v)+n'/f'=n'/(l'-v'),$$
(3)

$$p'/p=(n/n')[(l'-v')/(l-v)].$$
(4)

Now what we need to do is to eliminate l and l' from equations (1), (3), and (4) in order to get a relationship between v and v'. This is easier to do if we work with the vergences

$$L = n/l,$$
$$L' = n'/l',$$
$$V = n/v,$$
$$V' = n'/v',$$

and the equivalent power

$$F = n'/f'.$$

Rewriting (1), (3), and (4) in terms of these quantities gives

$$L'= L+F,$$
(5)

$$V'L'/(L'-V')=VL/(L-V)-F,$$
(6)

$$p'/p=[VL(L'-V')]/[V'L'(L-V)].$$
(7)

Now we must get an equation relating V and V' which doesn't involve L or L'.

Start by subtracting (5) from (6) to get

$$V'L'/(L'-V')+L'=VL/(L-V)+L.$$

Rationalizing and collecting terms we get

$$(L'-V')/(L-V)=L'^2/L^2.$$

Using (7) this becomes

$$L/L'=pV/p'V', \tag{8}$$

or,

$$m=(p/p')(n/n')(v'/v). \tag{9}$$

Equation (9) is the generalization of (2). Now we must find the generalization of (1).

Rationalize (6) to get

$$VL(L'-V')-F(L-V)(L'-V')+V'L'(L-V)=0.$$

Collecting terms gives,

$$[L+F-L']VV'-VL'[L+F]-V'L[F-L']+FLL'=0.$$

Applying (5) to each of the terms in square brackets gives

$$-VL'^2+V'L^2+FLL'=0.$$

Dividing this by LL' and applying (8) gives, after some simplification,

$$p'^2V'=p^2V+pp'F.$$

From the definition of V, V', and F,

$$p'^2(n'/v')=p^2(n/v)+pp'(n'/f'). \tag{10}$$

This is the appropriate generalization of (1).

IMAGE ILLUMINANCE IN WIDE APERTURE SYSTEMS

Let's again find the illuminance of an axial image, but without the small angle approximation and for a general optical system like that of figure 2. The flux through the entrance pupil is given by[1],

$$F = LS \int_{\text{entrance pupl}} \cos\theta \, d\omega = LS \int \frac{(\cos\theta)^2}{r^2} dA.$$

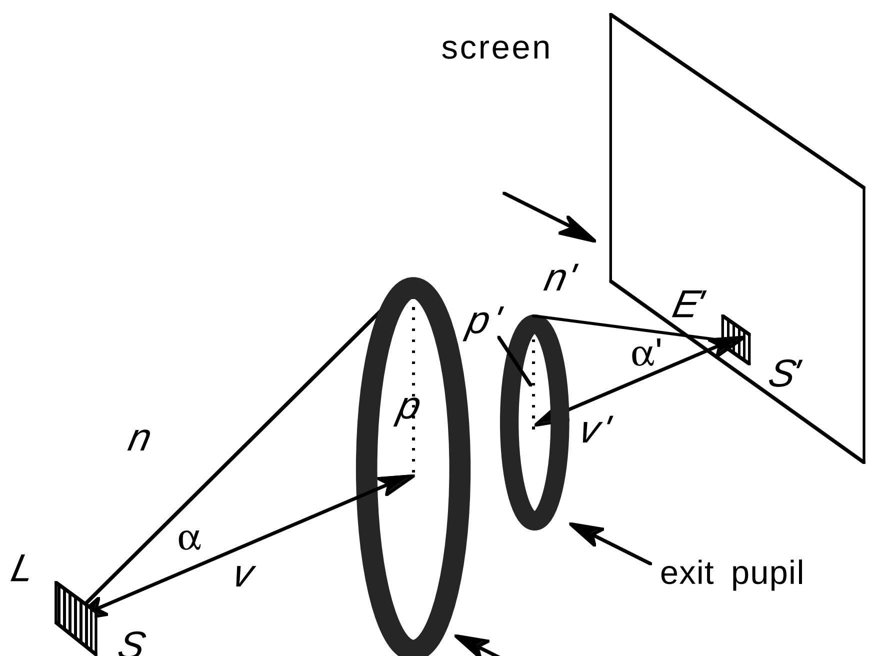

Figure 2. The illuminance of an image by a general optical system.

Let's do the integral precisely, without any assumptions about object position or pupil size. To do so, break the entrance pupil up into a series of concentric rings, each with inside radius ρ and outside radius ρ+dρ. The area of each ring and the distance from the object to the ring are, from the geometry of figure 3,

$$dA = 2\pi\rho d\rho,$$
$$r^2 = \rho^2 + v^2,$$
$$\cos\theta = \frac{v}{\sqrt{\rho^2 + v^2}},$$

and the integral becomes

$$\int_0^{p} \frac{(\cos\theta)^2}{r^2} \mathrm{d}A = 2\pi v^2 \int_0^{p} \frac{\rho}{\left(\rho^2 + v^2\right)^2} \mathrm{d}\rho.$$

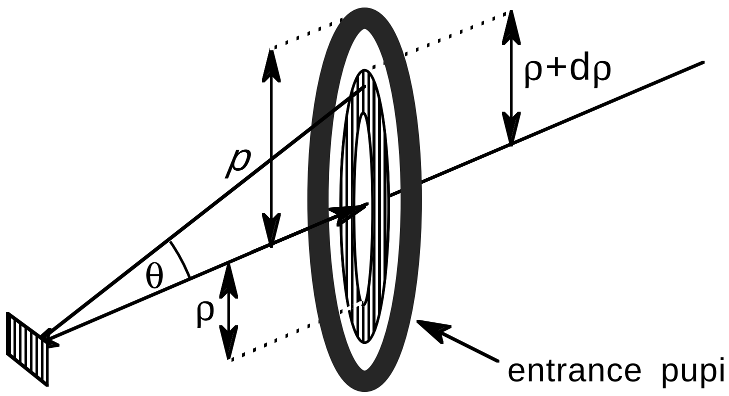

Figure 3. The entrance pupil is broken up into infinitesimally thin rings to calculate the total flux entering the optical system.

Evaluating this integral gives

$$\int[\rho/(\rho^2+v^2)^2]\mathrm{d}\rho = p^2/[2v^2(v^2+p^2)],$$

so that

$$\int(\cos^2\theta/r^2)\mathrm{d}A = \pi p^2/(v^2+p^2).$$

The flux into the system is, then

$$F = \pi L S p^2/(v^2+p^2).$$

The flux leaving the system is τF where τ is the transmittance of the optical system, and noting that the ratio of the image area to the object area is just the square of the magnification, the image illuminance is

$$E' = F'/S' = \tau F/S' = (\pi\tau L/m^2)[p^2/(v^2+p^2)].$$

From geometry, the quantity in brackets is $\sin^2\alpha$ so that

$$E'=\pi\tau L\sin^2\alpha/m^2. \tag{11}$$

Equation (11) is the equation for illuminance of the image of a wide angle, thick lens system.

The <u>numerical aperture</u> of an optical system, abbreviated N.A. is defined as

$$\text{N.A.}=n\sin\alpha,$$

where n is the index of object space. Equation (11) is sometimes written in terms of numerical aperture as

$$E'=\pi\tau L[(\text{N.A.})/(nm)]^2.$$

Finally, consider the case when the object is a long way from the entrance pupil so that $v\Rightarrow-\infty$. In this case

$$E'=\pi\tau L\sin^2\alpha/m^2=\pi\tau L[(p/v)/(p/p')(n/n')(v'/v)]^2=\pi\tau L(p'/v')^2(n'/n)^2,$$

where we've used equation (9). But from equation (10), when $v\Rightarrow-\infty$, $p'/v'\Rightarrow p/f'$, and E' becomes

$$E'=\pi\tau L(n'/n)^2(p/f')^2$$

or

$$E'=\frac{(\pi\tau L)(n'/n)^2}{4(\text{f/number})^2}$$

where f/number is the focal length divided by entrance pupil diameter.

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[1]See image photometry...paraxial, pages 1-2.