SOLID ANGLES

CONSTRUCTION OF A SOLID ANGLE

Before going on to a discussion of light sources it's necessary to make a mathematical detour in order to discuss a geometric entity, the solid angle.

First let's review, with the help of figure 1, the radian measure of angles in two dimensions. To find the angle subtended in radians by a line segment C_1C_2 at a point P, construct a circle of radius 1, a unit circle, around P. Next measure the length of the circular arc A_1A_2 of the unit circle cut off by the straight lines C_1P and C_2P. The length of the arc A_1A_2 equals the number of radians subtended by C_1C_2. It is important that the arc length A_1A_2 be measured in the same units as the radius of the unit circle. For example, if the circle has a radius of one meter and the arc length is 0.l meters, the angle is 0.1 radians. If an arc completely surrounds P it subtends 2π radians, the circumference of the unit circle. Note that the radian, being the quotient of two lengths, is a unitless quantity.

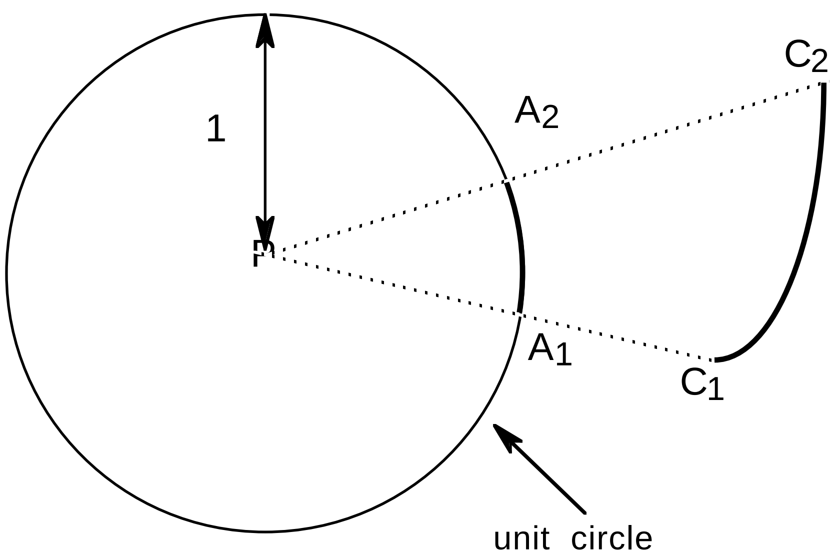

Figure 1. In radian measure the subtense of the arc C_1C_2 at point P is the length of arc A_1A_2 on the unit circle.

Now let's go to the three dimensional case. Consider a point P and a surface of area *A* somewhere in space. We want a measure of the subtense of the area at P. Taking our cue from the two dimensional case, construct a unit sphere, a sphere of radius 1, around P. The subtense of *A* is measured by the area it cuts out of the unit sphere, as illustrated in figure 2. This is the area which would be cut out by lines drawn from P to every point on the periphery of *A*. This area is the solid angle subtended by *A*. The unit of measurement of the solid angle is the steradian, abbreviated str, the three dimensional analog of the radian. For example, if the unit sphere has a one meter radius and *A* cuts out an area of 6 m^2 on the unit sphere, *A* subtends a solid angle of 6 steradians. The usual symbol for solid angle is ω.

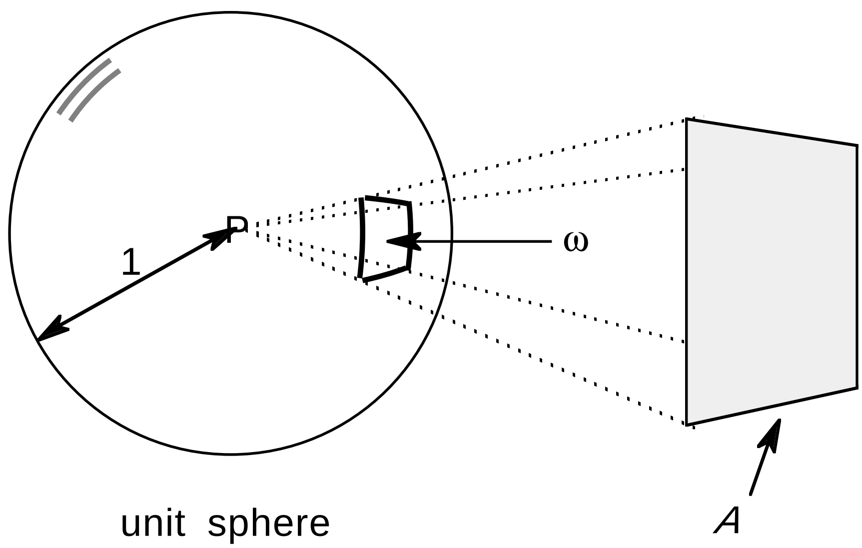

Figure 2. The solid angle subtended by area *A* at point P is measured by the area ω on the surface of the unit sphere centered at P.

CALCULATION OF SOLID ANGLE

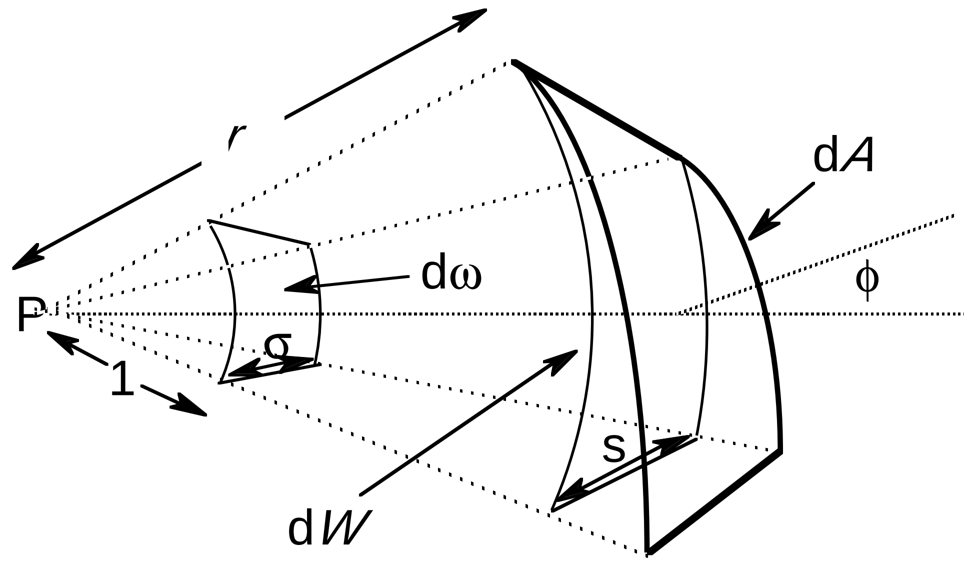

Figure 3. An infinitesimal area dA subtends an angle dω at point P. (The scale is much exaggerated!)

Let's calculate the solid angle dω subtended by an infinitesimal area dA at a point P. Since the area is infinitesimal, all points on A are essentially equidistant from P. Designate the distance from P to dA by r. Because the surface is so small, it may be considered essentially flat and a single angle will suffice to specify the orientation of dA. Let that angle be ϕ, the angle between the normal to the surface and the line connecting the surface and point P. This situation is shown in figure 3. First let's project dA onto the sphere of radius r centered at P. The area of that projection is $\cos\phi\,dA \equiv dW$.

The areas dW and dω have the same general shape, so if we take a typical dimension of each area, say the distance along one edge, $dW \propto s^2$ and $d\omega \propto \sigma^2$, where s and σ are the lengths of the typical dimension on dW and dω, respectively. Hence we have the proportionality

$$dW/d\omega = s^2/\sigma^2.$$

Looking at similar triangles in figure 3 in the plane containing r we arrive at the ratio,

$$s/\sigma = r/1$$

since 1 is, of course, the radius of the unit sphere. Combining these last two equations we get

$$dW/d\omega = (r/1)^2 = r^2,$$

or simply

$$d\omega = dW/r^2.$$

But since $dW = \cos\phi\, dA$, we obtain our final result,

$$d\omega = (\cos\phi/r^2)\,dA. \quad (1)$$

To find the solid angle subtended by a surface of finite size we have to integrate (1), obtaining

$$\omega = \int_A d\omega = \int_A (\cos\phi/r^2)\,dA.$$

We'll perform the integral for three cases.

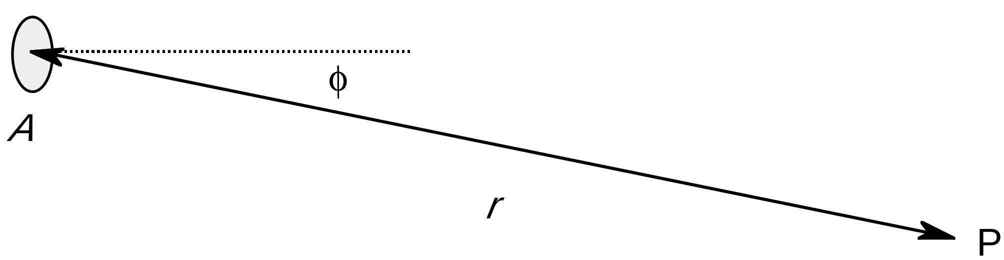

Figure 4. Subtense of a small area at a point.

The first case is that of a small, flat area with linear dimensions$<<r$. In this case the distance from P to any part of the area is about the same so r is effectively constant over the area. Likewise ϕ is effectively constant over the area, so

$$\omega = \int d\omega = \int (\cos\phi/r^2)\,dA = (\cos\phi/r^2)\int dA$$

or

$$\omega = A\cos\phi/r^2 \text{ ... for a small area.}$$

Another important case is the angle subtended by an area on a sphere of radius R at the center of the sphere (figure 5). In this case $\phi=0$ for any point on the surface and r equals the radius of the sphere for any point on the surface, hence the integral reduces to

$$\omega=\int d\omega=\int(\cos 0/R^2)dA=(1/R^2)\int dA$$

or simply

$$\omega=A/R^2 \text{ ... for an area on a sphere.}$$

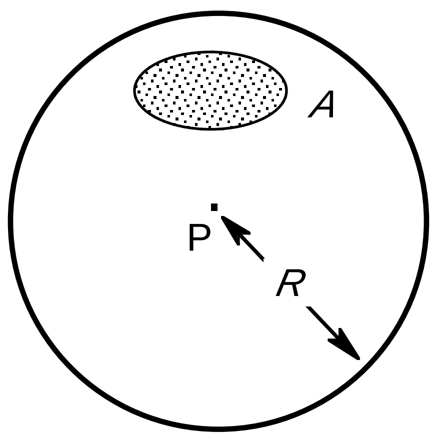

Figure 5. Subtense of an area on a sphere at its center.

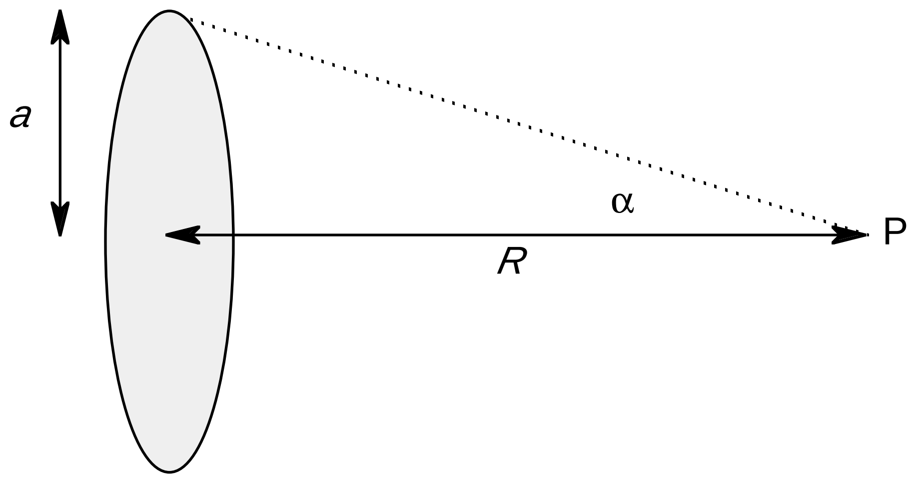

Figure 6. A disk of radius *a* subtending half angle α at a point P.

The final, and most complex case is that of the solid angle subtended by a disk at a point P on its axis, as shown in figure 6.

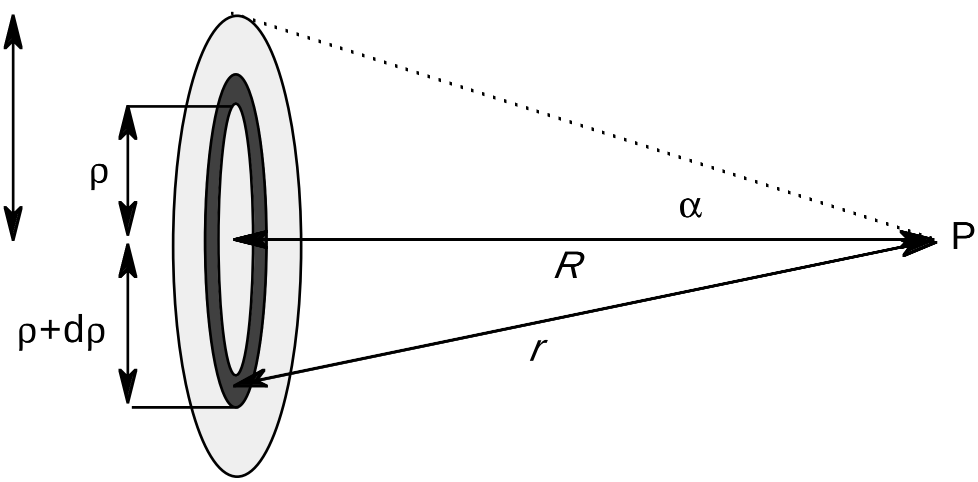

Figure 7. To calculate the subtense of a disk at point P, break the disk up into infinitesimally thin rings.

To apply (1), first break the circle up into a series of concentric ring shaped areas as in figure 7. The inside radius of each of these areas is ρ and the outside radius is $(\rho+d\rho)$ so that the area of each ring is

$$dA=\pi(\rho+d\rho)^2-\pi\rho^2=2\pi\rho d\rho,$$

where higher order terms in $d\rho$ have been dropped since $d\rho$ is infinitesimally small.

Any point on the ring is a distance

$$r=\sqrt{(R^2+\rho^2)}$$

from P and makes an angle

$$\phi=\cos^{-1}(R/r).$$

Substituting these relationships into (1) and integrating over the radius of the circle we get

$$\omega=\int d\omega=\int\frac{\cos\phi}{r^2}dA=\int\left[\frac{(R/r)}{r^2}\right]2\pi\rho d\rho=2\pi R\int_0^a\frac{\rho}{\left(R^2+\rho^2\right)^{3/2}}d\rho.$$

Evaluating the integral (how?) and simplifying,

$$\omega=2\pi\left[1-\frac{R}{\sqrt{R^2+a^2}}\right] \text{ ... for a disk.} \qquad (2)$$

If we write (2) in terms of α, the half angle subtended by the circle at P, it takes the simple form

$$\omega=2\pi(1-\cos\alpha) \text{ ... for a disk.} \qquad (3)$$

For a remote object α is small and so

$$\cos\alpha\cong 1-\alpha^2/2,$$

hence

$$\omega\cong\pi\alpha^2 \text{ ... for a small disk.} \qquad (4)$$

Note that α must be given in radians. (Why?)