Before going on to a discussion of light sources it's necessary to make a mathematical detour in order to discuss a geometric entity, the solid angle.

First let's review, with the help of figure 1, the radian measure of angles in two dimensions. To find the angle subtended in radians by a line segment $C_1C_2$ at a point $P$, construct a circle of radius 1, a unit circle, around $P$. Next measure the length of the circular arc $A_1A_2$ of the unit circle cut off by the straight lines $C_1P$ and $C_2P$. The length of the arc $A_1A_2$ equals the number of radians subtended by $C_1C_2$. It is important that the arc length $A_1A_2$ be measured in the same units as the radius of the unit circle. For example, if the circle has a radius of one meter and the arc length is 0.1 meters, the angle is 0.1 radians. If an arc completely surrounds $P$ it subtends $2\pi$ radians, the circumference of the unit circle. Note that the radian, being the quotient of two lengths, is a unitless quantity.
Now let's go to the three dimensional case. Consider a point $P$ and a surface of area $A$ somewhere in space. We want a measure of the subtense of the area at $P$. Taking our cue from the two dimensional case, construct a unit sphere, a sphere of radius 1, around $P$. The subtense of $A$ is measured by the area it cuts out of the unit sphere, as illustrated in figure 2. This is the area which would be cut out by lines drawn from $P$ to every point on the periphery of $A$. This area is the solid angle subtended by $A$. The unit of measurement of the solid angle is the steradian, abbreviated \textit{str}, the three dimensional analog of the radian. For example, if the unit sphere has a one meter radius and $A$ cuts out an area of $6 \text{ m}^2$ on the unit sphere, $A$ subtends a solid angle of $6$ steradians. The usual symbol for solid angle is $\omega$.

![Figure 2](image.png)

Figure 2. The solid angle subtended by area $A$ at point $P$ is measured by the area $\omega$ on the surface of the unit sphere centered at $P$. 
Let's calculate the solid angle $d\omega$ subtended by an infinitesimal area $dA$ at a point $P$. Since the area is infinitesimal, all points on $A$ are essentially equidistant from $P$. Designate the distance from $P$ to $dA$ by $r$. Because the surface is so small, it may be considered essentially flat and a single angle will suffice to specify the orientation of $dA$. Let that angle be $\phi$, the angle between the normal to the surface and the line connecting the surface and point $P$. This situation is shown in figure 3. First let's project $dA$ onto the sphere of radius $r$ centered at $P$. The area of that projection is $\cos \phi dA \equiv dW$.

The areas $dW$ and $d\omega$ have the same general shape, so if we take a typical dimension of each area, say the distance along one edge, $dW \propto s^2$ and $d\omega \propto \sigma^2$, where $s$ and $\sigma$ are the lengths of the typical dimension on $dW$ and $d\omega$, respectively. Hence we have the proportionality

$$dW/d\omega = s^2/\sigma^2.$$ 

Looking at similar triangles in figure 3 in the plane containing $r$ we arrive at
the ratio, \[ s/\sigma = r/1 \]
since 1 is, of course, the radius of the unit sphere. Combining these last two equations we get
\[ dW/d\omega = (r/1)^2 = r^2, \]
or simply
\[ d\omega = dW/r^2. \]
But since \( dW = \cos \phi \, dA \), we obtain our final result,
\[ d\omega = (\cos \phi / r^2) \, dA. \]
(1)

To find the solid angle subtended by a surface of finite size we have to integrate (1), obtaining
\[ \omega = \int d\omega = \int (\cos \phi / r^2) \, dA. \]

We'll perform the integral for three cases.

![Diagram](image)

Figure 4. Subtense of a small area at a point.

The first case is that of a small, flat area with linear dimensions \( << r \). In this case the distance from \( P \) to any part of the area is about the same so \( r \) is effectively constant over the area. Likewise \( \phi \) is effectively constant over the area, so
\[ \omega = \int d\omega = \int (\cos \phi / r^2) \, dA = (\cos \phi / r^2) \int dA \]
or
\[ \omega = A \cos \phi / r^2 \ldots \text{for a small area.} \]
Another important case is the angle subtended by an area on a sphere of radius $R$ at the center of the sphere (figure 5). In this case $\phi=0$ for any point on the surface and $r$ equals the radius of the sphere for any point on the surface, hence the integral reduces to

$$\omega = \int d\omega = \int (\cos 0 / R^2) dA = (1 / R^2) \int dA$$

or simply

$$\omega = A / R^2 \quad \text{... for an area on a sphere.}$$

Figure 5. Subtense of an area on a sphere at its center.
The final, and most complex case is that of the solid angle subtended by a disk at a point $P$ on its axis, as shown in figure 6.

Figure 7. To calculate the subtense of a disk at point $P$, break the disk up into infinitesimally thin rings.

To apply (1), first break the circle up into a series of concentric ring shaped areas as in figure 7. The inside radius of each of these areas is $\rho$ and the outside radius is $(\rho + \mathrm{d}\rho)$ so that the area of each ring is

$$\mathrm{d}A = \pi (\rho + \mathrm{d}\rho)^2 - \pi \rho^2 = 2\pi \rho \mathrm{d}\rho,$$

where higher order terms in $\mathrm{d}\rho$ have been dropped since $\mathrm{d}\rho$ is infinitesimally small.
Any point on the ring is a distance
\[ r = \sqrt{R^2 + \rho^2} \]
from P and makes an angle
\[ \phi = \cos^{-1}(R/r). \]

Substituting these relationships into (1) and integrating over the radius of the circle we get

\[ \omega = \int d\omega = \int \frac{\cos \phi}{r^2} dA = \int \left[ \frac{(R/r)}{r^2} \right] 2\pi \rho d\phi = 2\pi R \int_0^a \frac{\rho}{\left( R^2 + \rho^2 \right)^{3/2}} d\rho. \]

Evaluating the integral (how?) and simplifying,

\[ \omega = 2\pi \left[ 1 - \frac{R}{\sqrt{R^2 + a^2}} \right] \quad \text{... for a disk.} \]

(2)

If we write (2) in terms of \( \alpha \), the half angle subtended by the circle at P, it takes the simple form

\[ \omega = 2\pi(1 - \cos \alpha) \quad \text{... for a disk.} \]

(3)

For a remote object \( \alpha \) is small and so

\[ \cos \alpha \equiv 1 - \frac{\alpha^2}{2}, \]

hence

\[ \omega \equiv \pi \alpha^2 \quad \text{... for a small disk.} \]

(4)

Note that \( \alpha \) must be given in radians. (Why?)