WAVES

GENERAL WAVE FUNCTION

A wave function is one which moves uniformly with time while retaining its shape. The general functional dependence of a wave on position, x, and time, t, is

$$y(x,t)=y(x\pm vt)$$

where v is the speed of the wave.

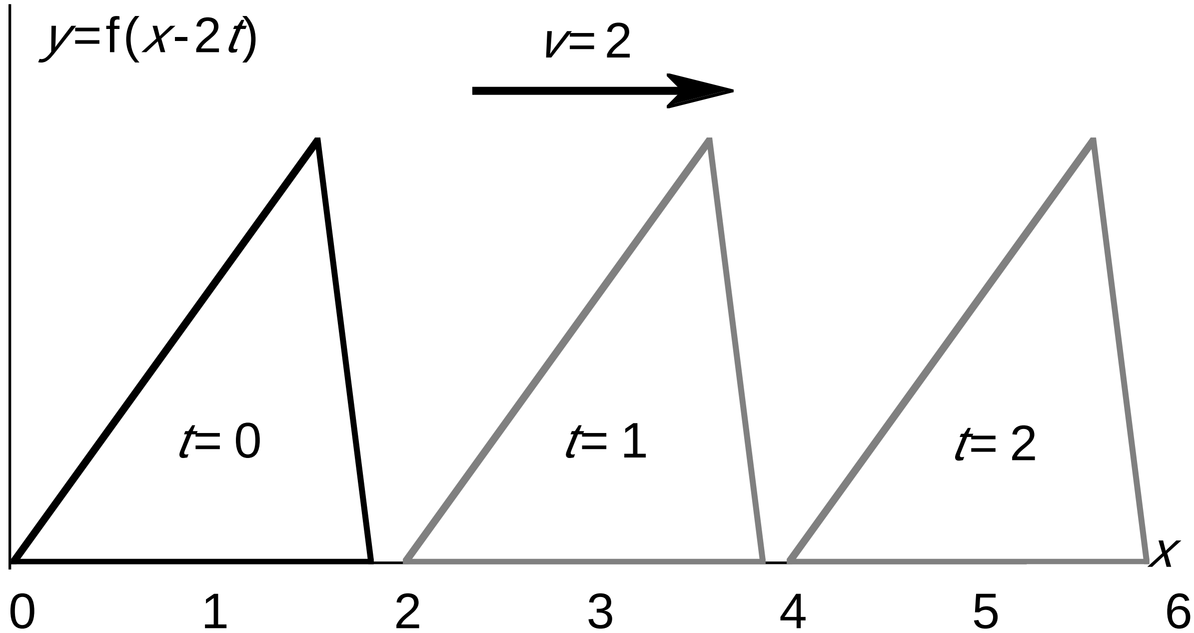

Figure 1. The sawtooth wave moves to the right without changing shape.

Figure 1 shows a saw-tooth wave moving to the right at speed v=2 at three different times, t. Note that the shape of the wave doesn't change as the wave moves.

SINE WAVES

The simplest wave function to deal with is the sine wave like that of figure 2. Its general form is

$$y(x,t)=a\sin\{[2\pi(x/\lambda-t/T)+\varepsilon]\}=a\sin\{[2\pi(sx-\nu t)+\varepsilon]\}$$
$$=a\sin(kx-\omega t+\varepsilon)$$

where we have the following definitions:

a = amplitude
λ = wavelength
s = spatial frequency $\equiv 1/\lambda$
T = period
ν = frequency = $1/T$
v = speed of propagation = λ/T
k = propagation number $\equiv 2\pi/\lambda$
ω = angular frequency $\equiv 2\pi/T$
ε = phase factor

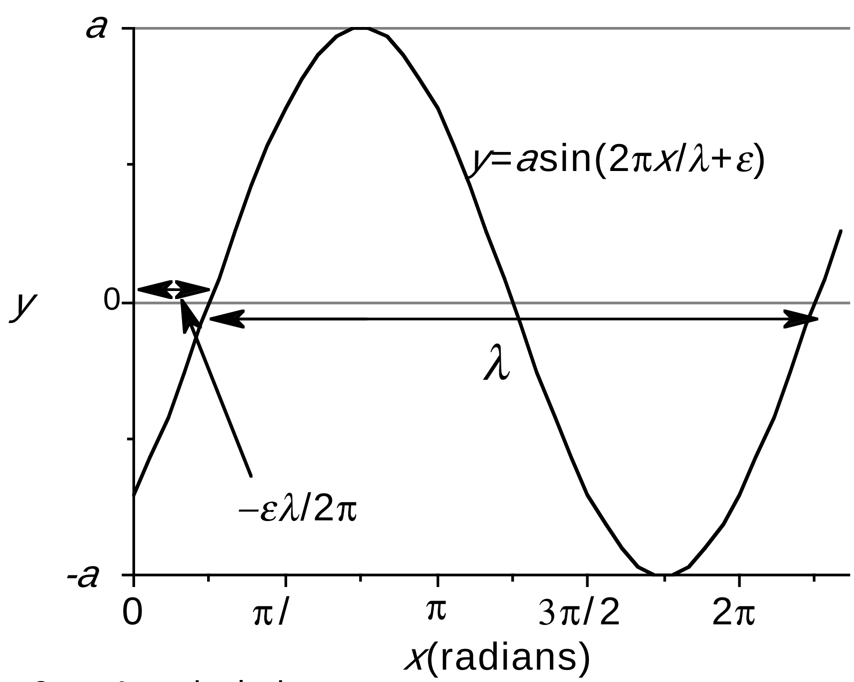

Figure 2. A typical sine wave when t=0.

The phase difference δ between two points on a wave separated by distance l is $\delta=k(x_2-x_1)=(2\pi/\lambda)l$.

ADDITION OF SINE WAVES

Suppose we have two sine functions, $y_1=a_1\sin(kx-\omega t+\varepsilon_1)$ and $y_2=a_2\sin(kx-\omega t+\varepsilon_2)$ and we want to add them to get $y=y_1+y_2$. How do we do it? There are three ways:

(1) Point-by-point addition

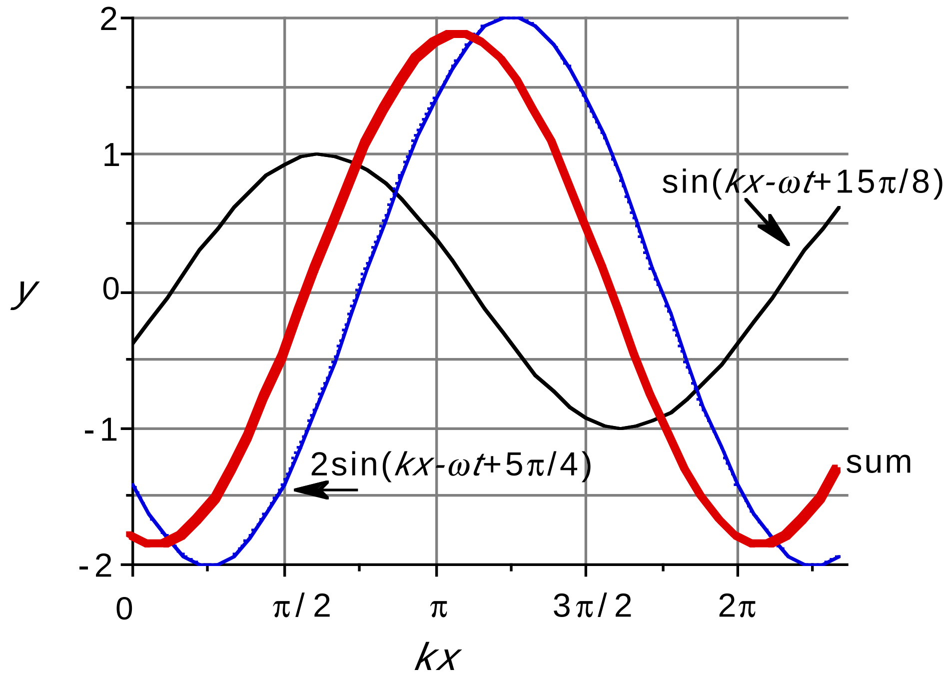

Figure 3. Adding sine waves point by point. The curves are plotted for t=0. At other times their position will be shifted, but their shapes will remain the same.

For example, in figure 3 when $kx=\pi$ (and t=0), $\sin(kx-\omega t+15\pi/8)=0.4$ and $2\sin(kx-\omega t+5\pi/4)=1.4$. Summing these gives 1.8, the value of the corresponding point on the dark line. Do this enough times and the entire dark line sum can be constructed. This is straightforward, but not very practical.

(2) Application of an equation derived from trigonometric identity

It is possible to solve this problem by manipulating trigonometric identities. When the sinusoids have the same wavelength and frequency, the result leads to the phasor method.

(3) Using phasor addition

A vector method is simplest, but it only works if both terms have the same argument (kx-ωt). Fortunately, this is always the case in physical optics. In that case the resultant is always a sine wave of argument (kx-ωt) as well.

The recipe for adding two waves $y_1 = a_1 \sin(kx - \omega t + \varepsilon_1)$ and $y_2 = a_2 \sin(kx - \omega t + \varepsilon_2)$ is to represent each wave as a two dimensional vector with length equal to the amplitude making an angle equal to its phase with the x axis. Such a vector is called a <u>phasor</u>. Then simply add these two vectors in the usual way and infer the phase and amplitude of the sum from the length and orientation of the resultant.

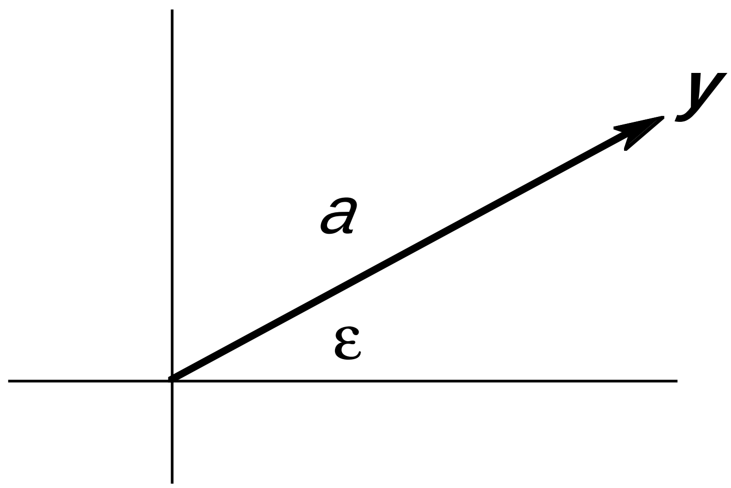

Figure 4. The vector $\boldsymbol{y}$ corresponding to the wave $y = a \sin(kx - \omega t + \varepsilon)$.

The figure shows the vector representation of the function $y = a \sin(kx - \omega t + \varepsilon)$. Let's apply vector addition to the problem we previously did with point-by-point addition.

Example: What function results from adding the waves $\sin(kx-\omega t+15\pi/8)$ and $2\sin(kx-\omega t+5\pi/4)$?

Solution: The vector representations of the two functions are added as in figure 5.

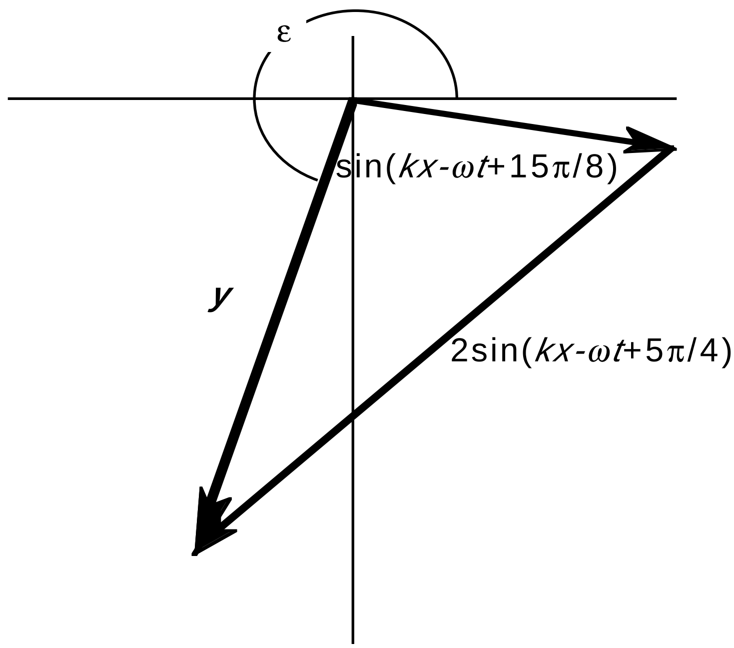

Figure 5. An example of adding two sine waves vectorially.

The components of the vector representing $\sin(kx-\omega t+15\pi/8)$ are $(1\cos 15\pi/8,\ 1\sin 15\pi/8)=(0.924,-0.383)$ and the components of the vector representing $2\sin(kx-\omega t+5\pi/4)$ are $(2\cos 5\pi/4,\ 2\sin 5\pi/4)=(-1.414,\ -1.414)$. Adding these together component by component gives $(-0.490,-1.797)=\boldsymbol{y}$, where $\boldsymbol{y}$ represents the sum of the two waves. The magnitude of $\boldsymbol{y}$ is

$$a=\sqrt{[(-.490)^2+(-1.797)^2]}=1.863$$

and the phase angle ε is given by[1]

$$\varepsilon=\tan^{-1}[(-1.797)/(-.490)]=4.446\ \text{rad}=1.415\pi\ \text{rad}.$$

Finally, then,

$$\sin(kx-\omega t+15\pi/8)+2\sin(kx-\omega t+5\pi/4)=1.863\sin(kx-\omega t+1.415\pi).$$

Look at figure 3 to see if this agrees with the point-by-point method.

ADDING WAVES WITH DIFFERENT FREQUENCIES

Physical optics generally deals with interference of sine waves of the same spatial and temporal frequency. But there are a couple of cases in which adding sine waves of different frequencies have important applications elsewhere in physical science or visual optics.

BEATS

In some branches of physics, notably acoustics, it is necessary to add sine waves of slightly different frequency. The resultant in such cases is, if the two component frequencies are close enough, a sine wave with periodically oscillating amplitude producing "beats". The frequency of the beats is the difference between the frequencies of the two interfering sine waves.

FOURIER THEOREM

In this course we'll always talk about sine waves. But what about more general wave forms? In effect we'll have disposed of them too because of the Fourier theorem.

The Fourier theorem states that every periodic function can be represented as an infinite sum of sine waves. If we know what happens to the individual sine wave components, we can determine what happens to the wave function through decomposition and then recomposition.

HUYGEN'S PRINCIPLE

Huygen's principle allows one to construct the position of a wavefront at a later time knowing its position at a given time. The idea is as follows:

1. Each point on a wavefront is regarded as a point source.

2. A spherical (circular in two dimensions) wave boils off these point sources.

3. The position of the final wave is the envelope of all these spherical waves.

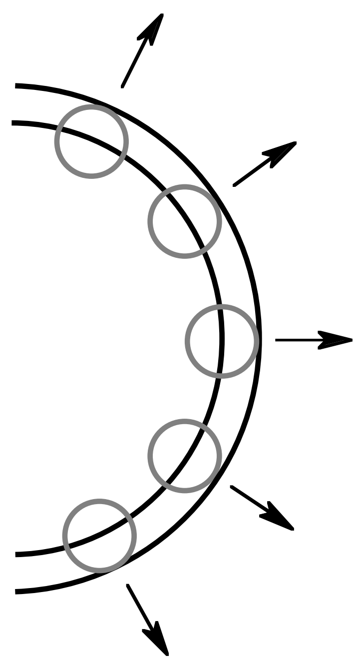

Figure 6. Huygen's construction.

Certain problems develop in application of the principle. Why, for example, is there no backscattering? The answer to this and other questions come from detailed application of electromagnetic theory.

Huygen's principle may be applied to a variety of problems. It can be used to develop Snell's law and law of reflection. More entertaining, though, is the explanation of the V-shaped wake of a duck.

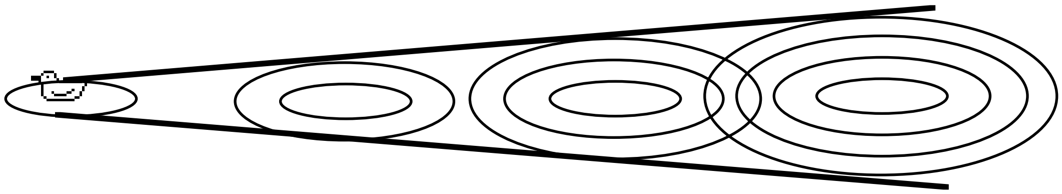

Figure 7. Constructing the wake of a swimming duck using Huygen's principle. The diagram is seen in perspective.

The wake of a swimming duck shown in figure 7 is explained by considering duck to be a chain of disturbances in water, and as the duck progresses each previous disturbance boils out to produce a V–shaped envelope. What happens if the duck accelerates, decelerates, or turns?

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[1]With an electronic calculator, one actually obtains $\alpha=\tan^{-1}[(-1.797)/(-0.490)]=1.305$ rad. This is because the calculator gives the principal value of the arc tangent, the angle that lies in the range $\pm\pi$. From the signs of the components in this case it is clear that α lies in the third quadrant so the calculator answer must be adjusted by adding π to get (1.305 rad+3.142 rad)=+4.446 rad.